Bifurcation and Exceptional Point Boundaries of Coupled Opto-Electronic Oscillators

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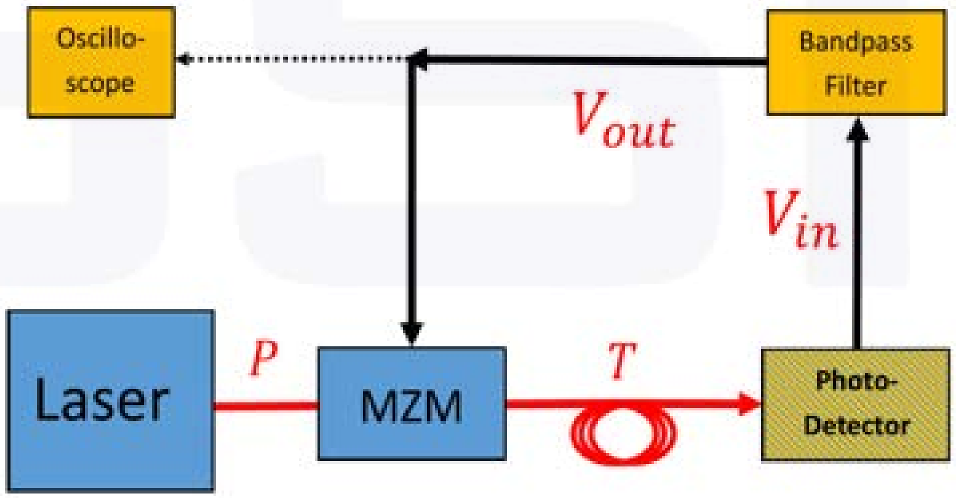
1. Intro to Opto-Electronic Oscillators (OEO)

- High-quality factor radio frequency (RF) signal generation
- RF Amplifier/Sensor (Linear and Non-linear regimes)
- Physical Random Number Generation
- Electronic ↔ Optical Conversion
- Well studied in experimental settings
- Understudied numerically and analytically
    - Generally hard to numerically model and analytically solve



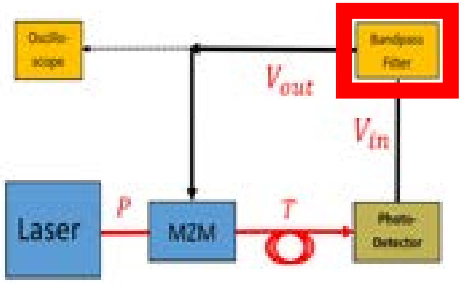
P: Laser Optical Power
V_{in}: Filter Input Voltage
T: Fiber-optic Delay Time
V_{out}: Filter Output Voltage

Why Analytic Solutions?

- Our group's longstanding interest in solving time-delayed differential equations
  - Coupled LASER models (Dr. Andrew Wilkey and Dr. Joseph Suelzer)
- Physical intuition & understanding
- Well-defined (and fast) experimental predictions

  → **Engineering devices** (ex: RF sensing)

2. OEO Model



LCR Filter

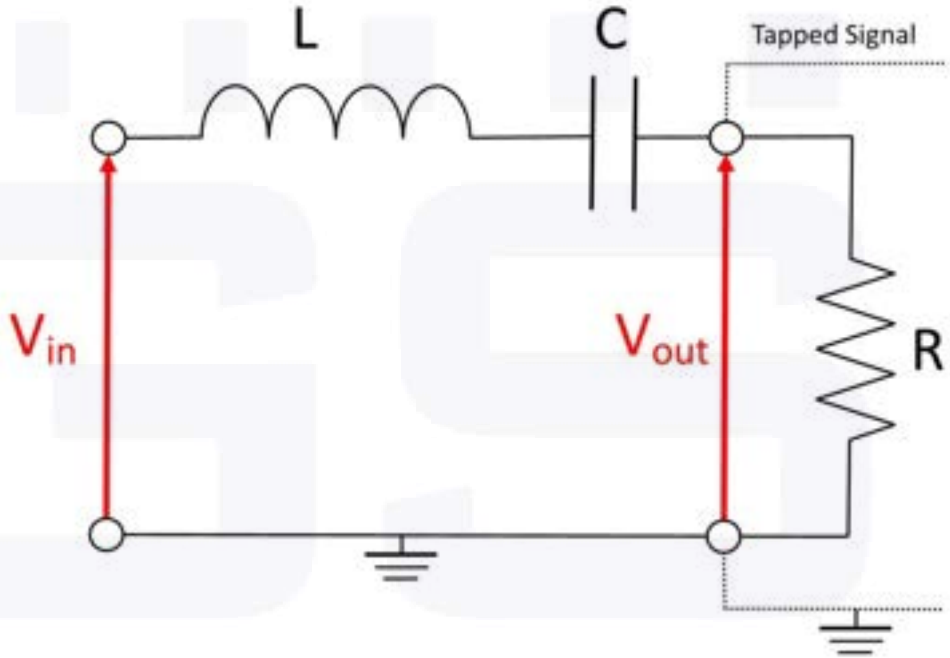
Using Kirchhoff's laws:

$$V_{out} + \frac{L}{R}\frac{dV_{out}}{dt} + \frac{1}{CR}\int_{t_0}^{t} V_{out} d\tau = V_{in}$$

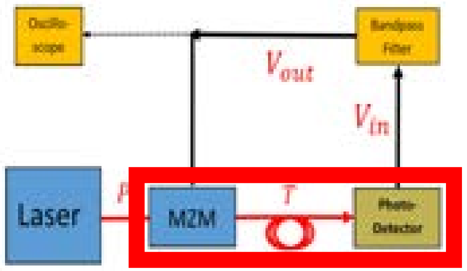
Defining:

$\Delta = \frac{R}{L}$, the (ang.) bandwidth

$\Omega_0 = \frac{1}{\sqrt{LC}}$, the central (ang.) frequency



$$\boxed{\frac{1}{\Delta}\frac{d^2V_{out}}{dt^2} + \frac{\Omega_0^2}{\Delta}V_{out} + \frac{dV_{out}}{dt} = \frac{dV_{in}}{dt}} \quad (1)$$



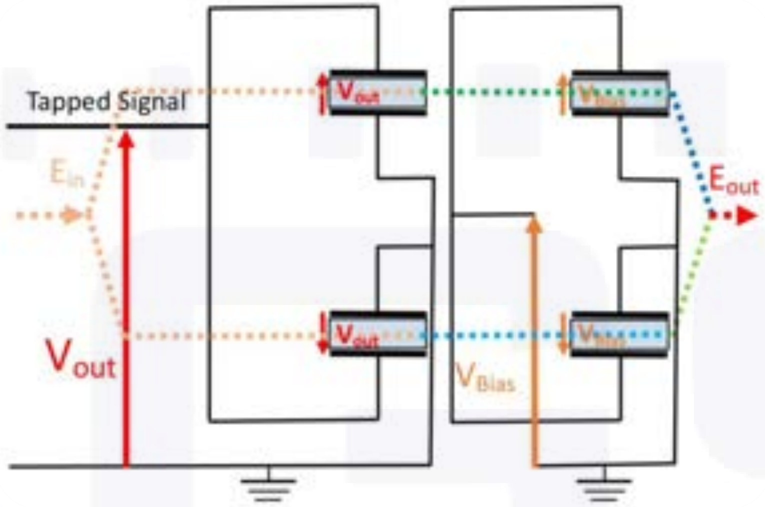
2. OEO Model

Mach-Zehnder Modulator (MZM) & Photo-Detector

MZM optical input (P) is:

1. beam-split
2. phase-shifted
3. recombined

$$P_D = P\cos^2(\kappa_s V_{out}(t) + \kappa_B V_{Bias})$$

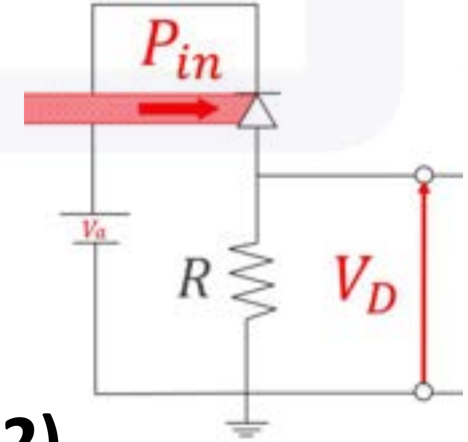


P - Laser Optical Power
P_D - MZM Output Power
V_{out} - Filter Output Voltage
κ_s - Input Voltage Coupling
V_{Bias} - MZM Bias Voltage
κ_B - Bias Voltage Coupling

Photo-detector construction:

- Resistor
- photo-diode
- Reverse Bias Voltage source

$$V_D = DP_{in} = DP_D(t - T)$$



P_{in} - Det. Input Power
V_D - Det. Output Voltage
V_a - Reverse Bias Voltage
R - Resistance

G - Integrated gain/loss

$$\boxed{V_{in} = GDP\cos^2[\kappa_s V_{out}(t - T) + \kappa_B V_{Bias}]} \quad \textbf{(2)}$$

2. OEO Model

$$\boxed{x = \frac{V_{out}}{\kappa_s},\ \phi = \frac{V_{Bias}}{\kappa_B},\ \alpha = \kappa_s DGP}$$

Model Equation

Plugging (2)

$$V_{in} = GDP\ cos^2[\kappa_s V_{out}(t-T) + \kappa_B V_{Bias}]$$

Into (1)

$$\frac{1}{\Delta}\frac{d^2 V_{out}}{dt^2} + \frac{\Omega_0^2}{\Delta} V_{out} + \frac{dV_{out}}{dt} = \frac{dV_{in}}{dt}$$

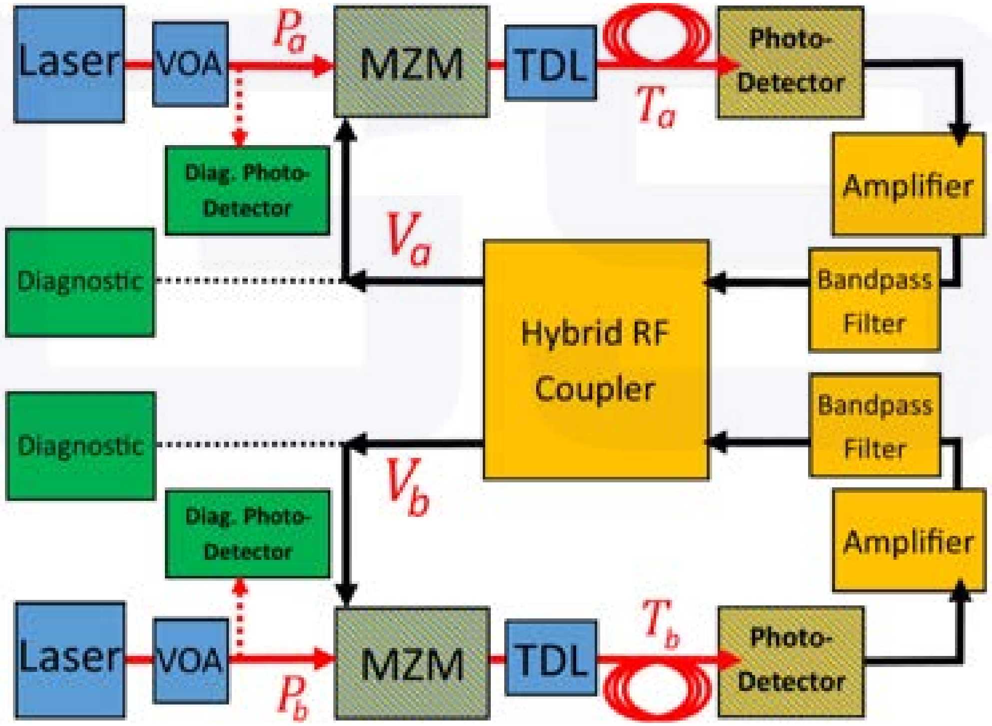
$$\boldsymbol{\frac{1}{\Delta}\frac{d^2 x}{dt^2} + \frac{\Omega_0^2}{\Delta} x + \frac{dx}{dt} = \alpha \frac{d}{dt} cos^2[x(t-T) + \phi]}$$

2. Coupled OEO Model

$$\frac{d^2 x_1}{dt^2} + \Omega_0^2 x_1 = \Delta \frac{d}{dt}\left(-x_1 + \alpha_1 \cos^2[\xi_a x_1(t - T_1) + \chi_a x_2(t - T_1) + \Phi_a]\right)$$

$$\frac{d^2 x_2}{dt^2} + \Omega_0^2 x_2 = \Delta \frac{d}{dt}\left(-x_2 + \alpha_2 \cos^2[\xi_b x_2(t - T_2) + \chi_b x_1(t - T_2) + \Phi_b]\right)$$

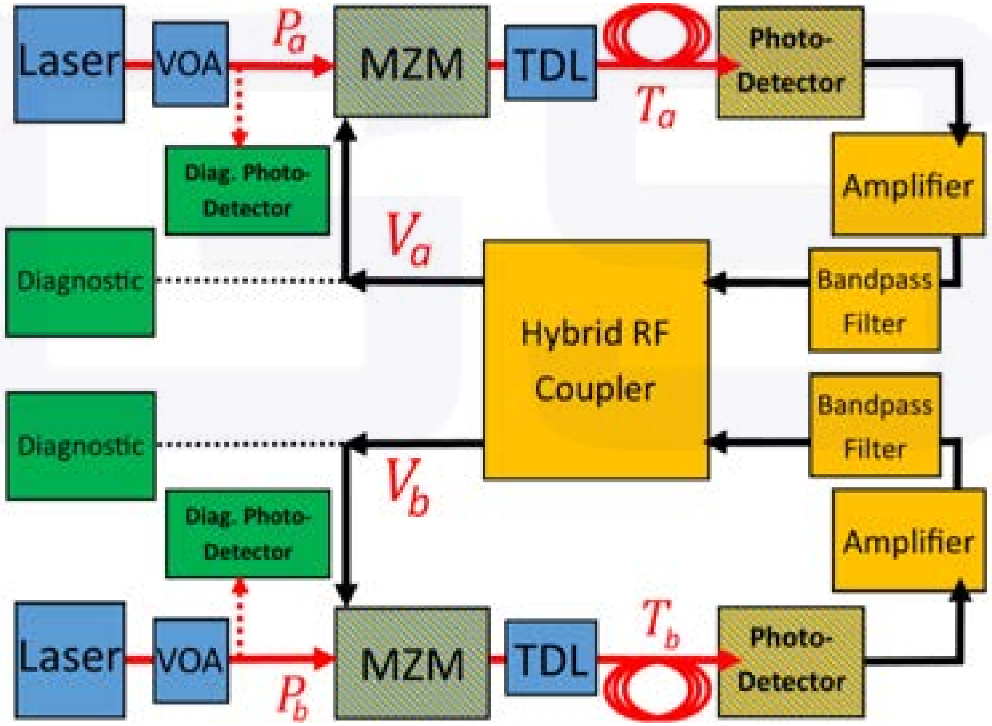
x_1, x_2 - Normalized Voltages
Ω_0 - Filter Central Frequency
T_1, T_2 - Time Delay
Δ - Bandwidth (Small)
α_1, α_2 - Effective Gain
ξ_a, ξ_b - Self Coupling
χ_a, χ_b - Cross Coupling
ϕ_a, ϕ_b - Normalized MZM Bias



2. Coupled OEO Model

$$\frac{d}{dt}\begin{bmatrix}A_1\\A_2\end{bmatrix} = -\frac{\Delta}{2}\left(\begin{bmatrix}\alpha_1\xi_a\sin(2\phi_a)\,e^{-i\Omega_0T_1-\tau\lambda} & \alpha_1\chi_b\sin(2\phi_a)e^{-i\Omega_0T_1-\tau\lambda}\\ \alpha_2\chi_a\sin(2\phi_b)e^{-i\Omega_0T_2-\tau\lambda} & \alpha_2\xi_b\sin(2\phi_b)e^{-i\Omega_0T_2-\tau\lambda}\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}A_1\\A_2\end{bmatrix}$$

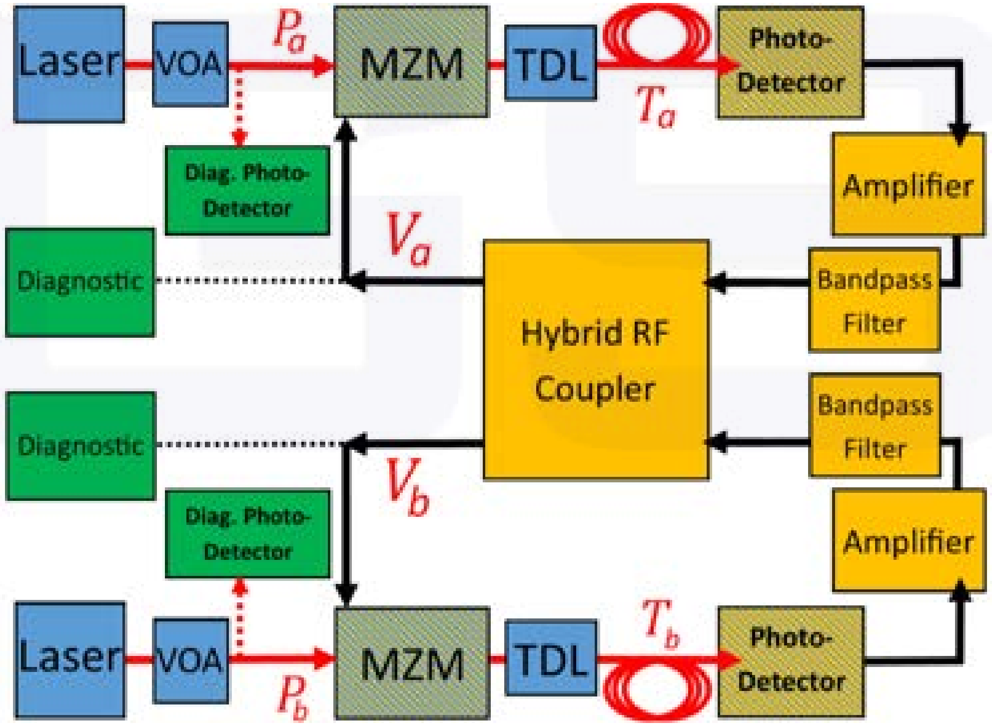
A_1, A_2 - Slowly Varying Amplitude
Ω_0 - Filter Central Frequency
T_1, T_2 - Fast Varying Carrier Time Delay
τ - Slow Varying Envelope Time Delay
Δ - Bandwidth (Small)
α_1, α_2 - Effective Gain
ξ_a, ξ_b - Self Coupling
χ_a, χ_b - Cross Coupling
ϕ_a, ϕ_b - Normalized MZM Bias



2. Coupled OEO Model

$$\frac{d}{dt}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = -\frac{\Delta}{2}\left(e^{i\theta_c}\begin{bmatrix} \alpha_1 e^{i\theta_d-\lambda\tau} & -\alpha_1 e^{i\theta_d-\lambda\tau} \\ \alpha_2 e^{-i\theta_d-\lambda\tau} & \alpha_2 e^{-i\theta_d-\lambda\tau} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

A_1, A_2 - Slowly Varying Amplitude
Ω_0 - Filter Central Frequency
$\Omega_0 T_1, \Omega_0 T_2 \Rightarrow -\theta_c - \theta_d, -\theta_c + \theta_d$
τ - Slow Varying Time Delay
Δ - Bandwidth
α_1, α_2 - Integrated Feedback Gain
$\xi_a, \xi_b \Rightarrow 1, 1$
$\chi_a, \chi_b \Rightarrow 1, -1$
$\phi_a, \phi_b \Rightarrow \frac{\pi}{4}, \frac{\pi}{4}$



2. Coupled OEO Model

Coordinates of threshold boundary:

For: $8\alpha_2\alpha_1 < (\alpha_1 + \alpha_2)^2$:

$$\alpha_2 = \frac{\alpha_1 - 1}{2\alpha_1 - 1}$$

$$\omega_{th} = 0$$

For: $8\alpha_2\alpha_1 > (\alpha_1 + \alpha_2)^2$:

$$\alpha_2 = (Long\ Expression)$$

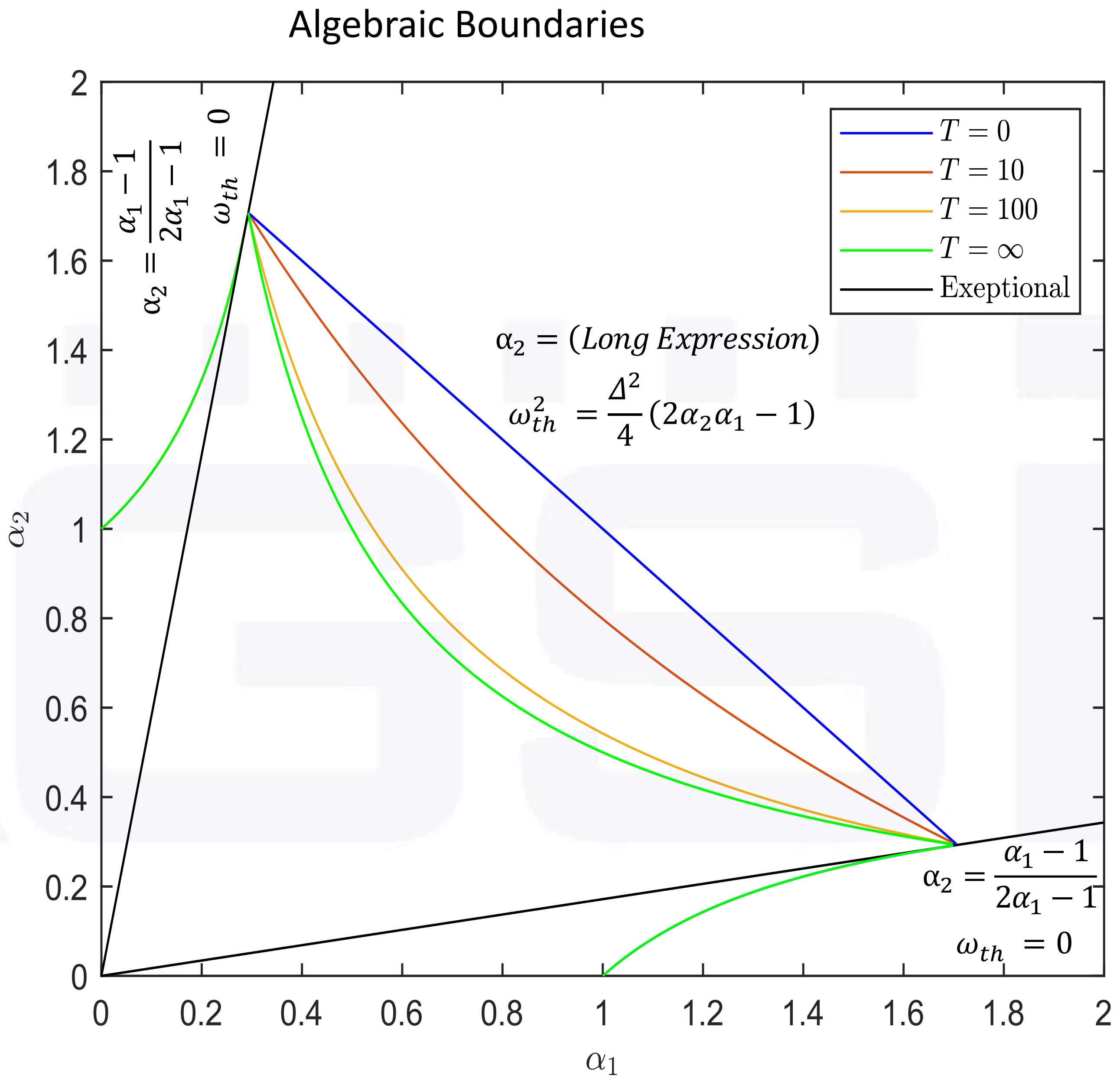
$$\omega_{th}^2 = \frac{\Delta^2}{4}(2\alpha_2\alpha_1 - 1)$$

****This result is valid for any time delay τ and but only for $\theta_c , \theta_d = 0$,** (Balanced Case)

2. COEO Model

Balanced Boundary

$$T = 0 \rightarrow \infty$$



$\alpha_1 \propto$ Optical Power 1

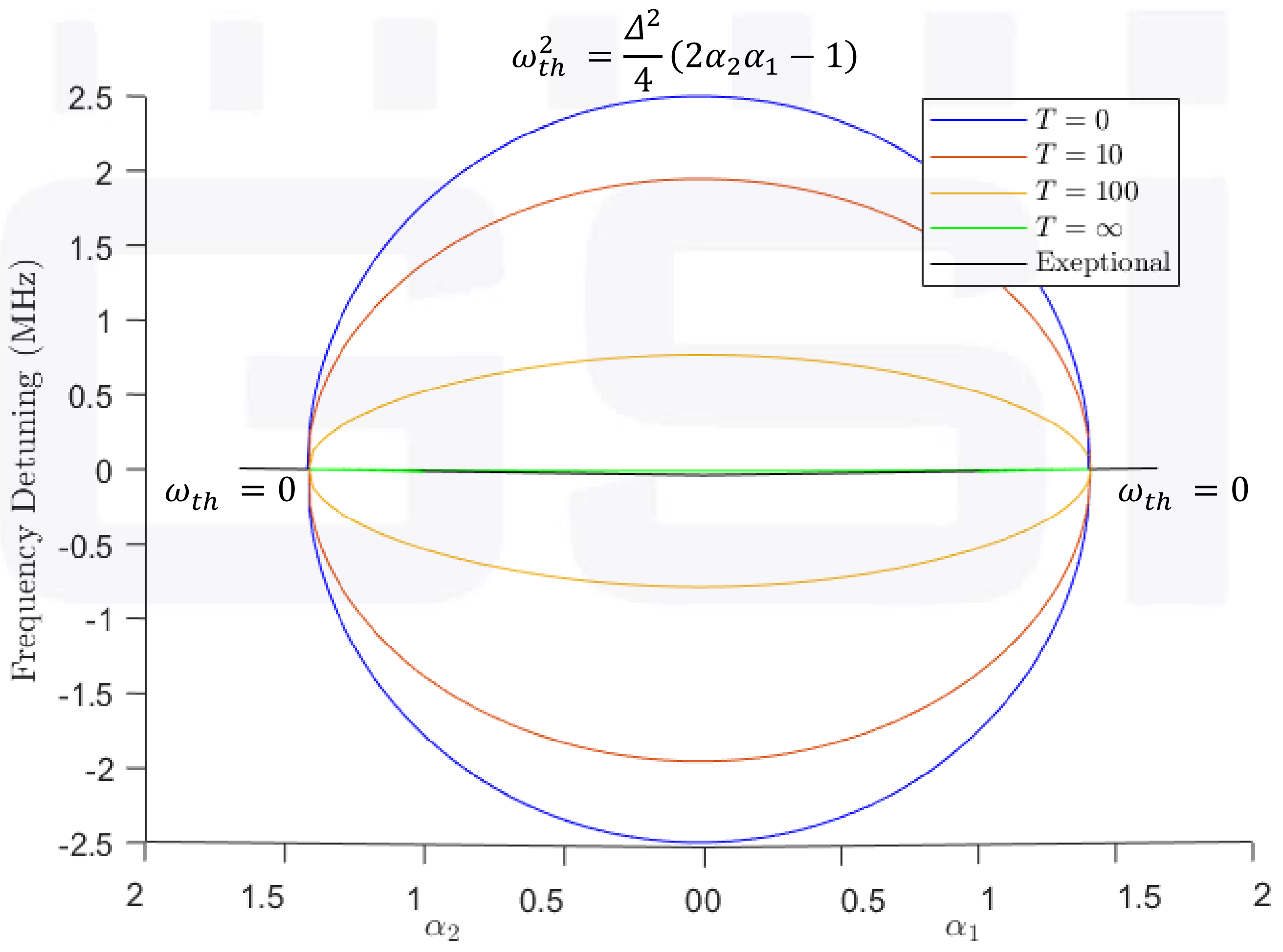
$\alpha_2 \propto$ Optical Power 2

Exceptional Boundary: $8\alpha_2\alpha_1 = (\alpha_1 + \alpha_2)^2$

2. COEO Model

Balanced Boundary

$$T = 0 \rightarrow \infty$$



$\alpha_1 \propto$ Optical Power 1
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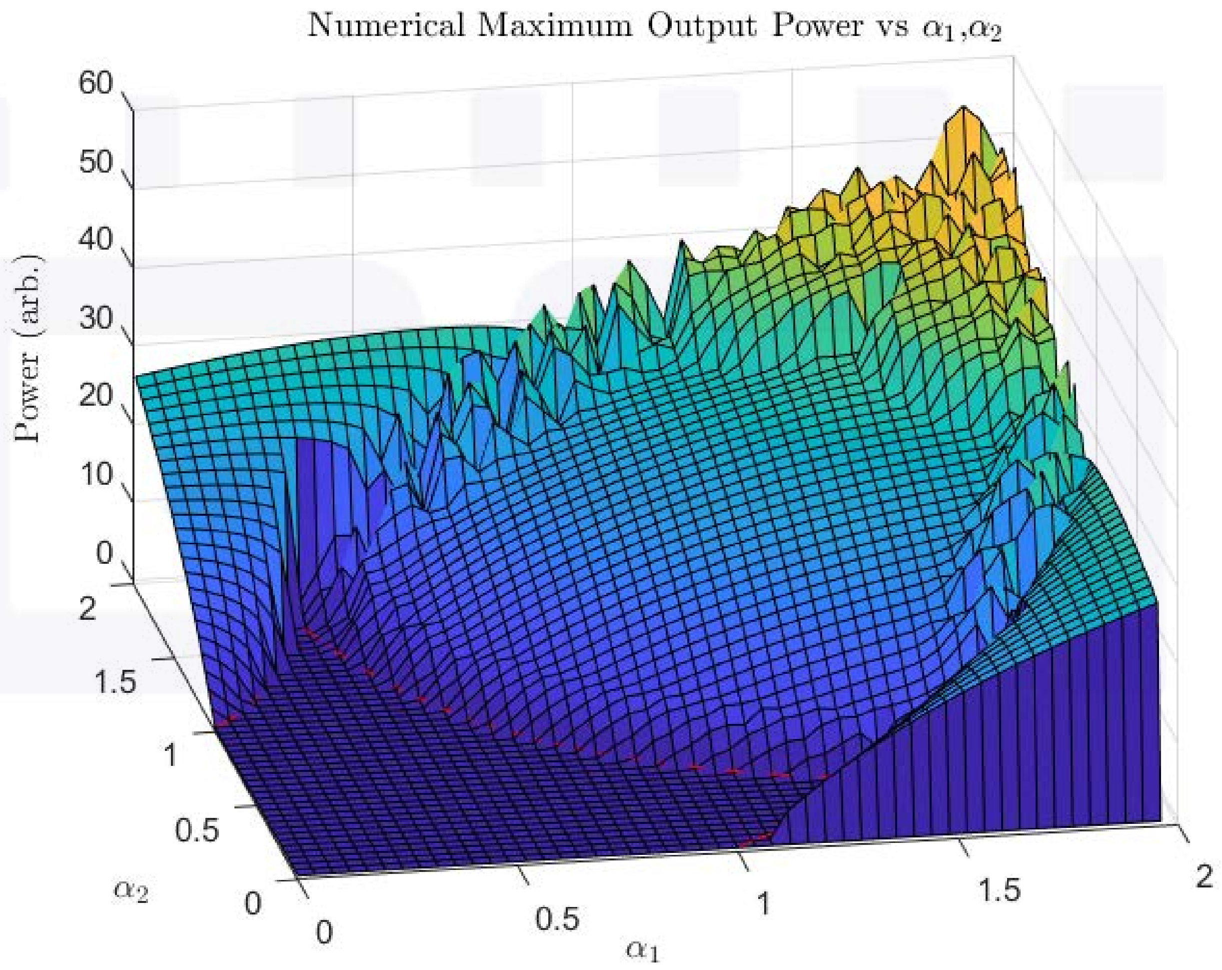
Exceptional Boundary: $8\alpha_2\alpha_1 = (\alpha_1 + \alpha_2)^2$

3. COEO Model

Numerical Agreement



$\alpha_1 \propto$ Optical Power 1
$\alpha_2 \propto$ Optical Power 2

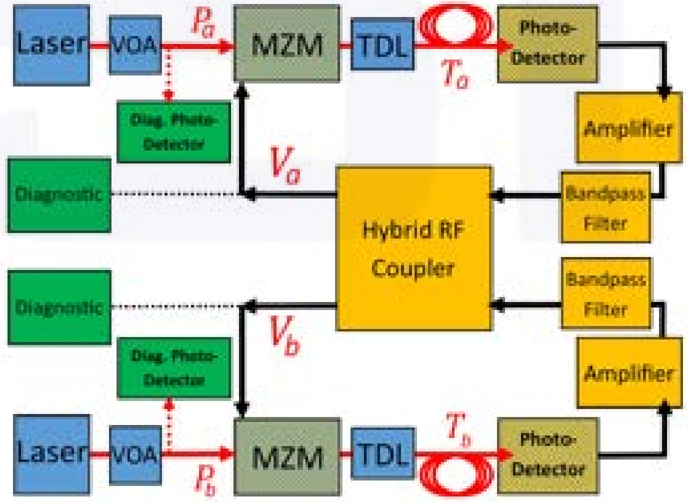
Exceptional Boundary: $8\alpha_2\alpha_1 = (\alpha_1 + \alpha_2)^2$

3. Experimental Results

3.1 Balanced Experiment

3.2 Phase Matched ($\Delta f_0(\theta_c) = 0.93$ MHz) Experiment

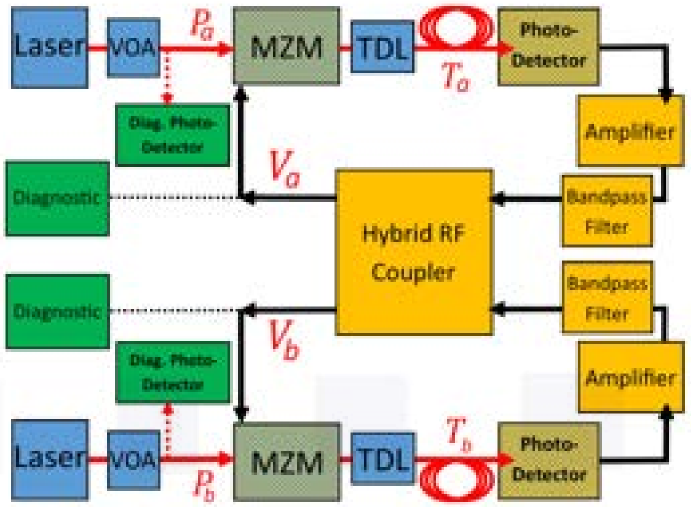
3.3 Frequency State Probability vs. Frequency Detuning



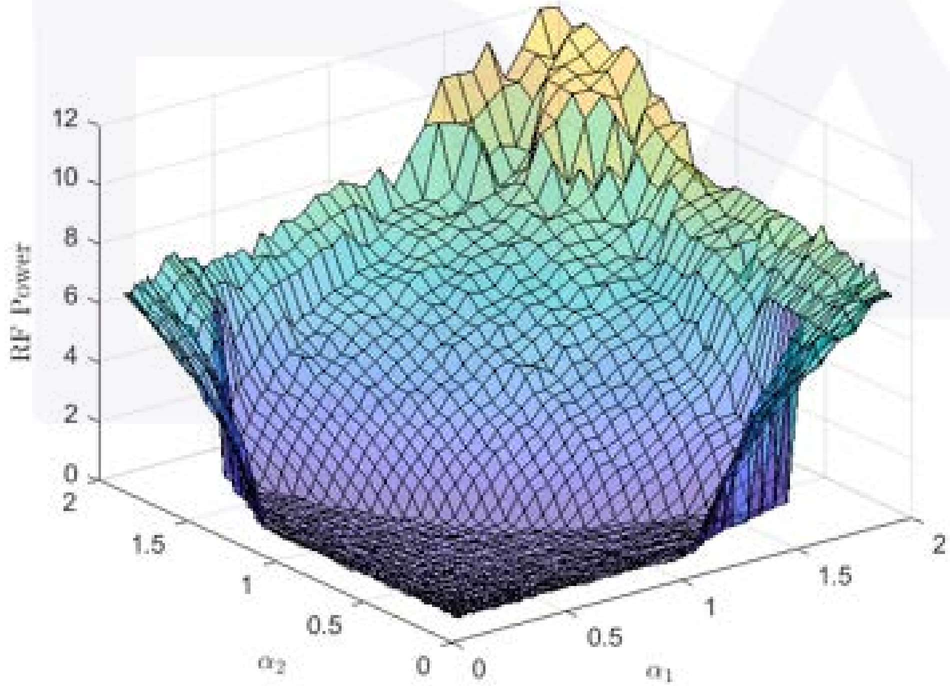
3. Exp. Results

Balanced

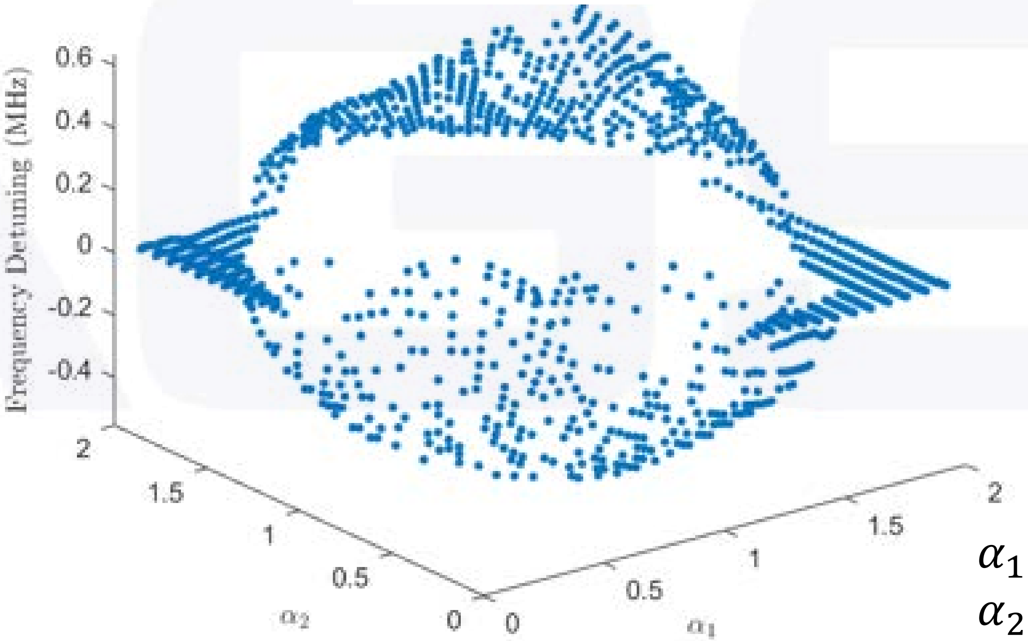
Experiment



Total RF Power



Frequency Detuning



$\alpha_1 \propto$ Optical Power 1

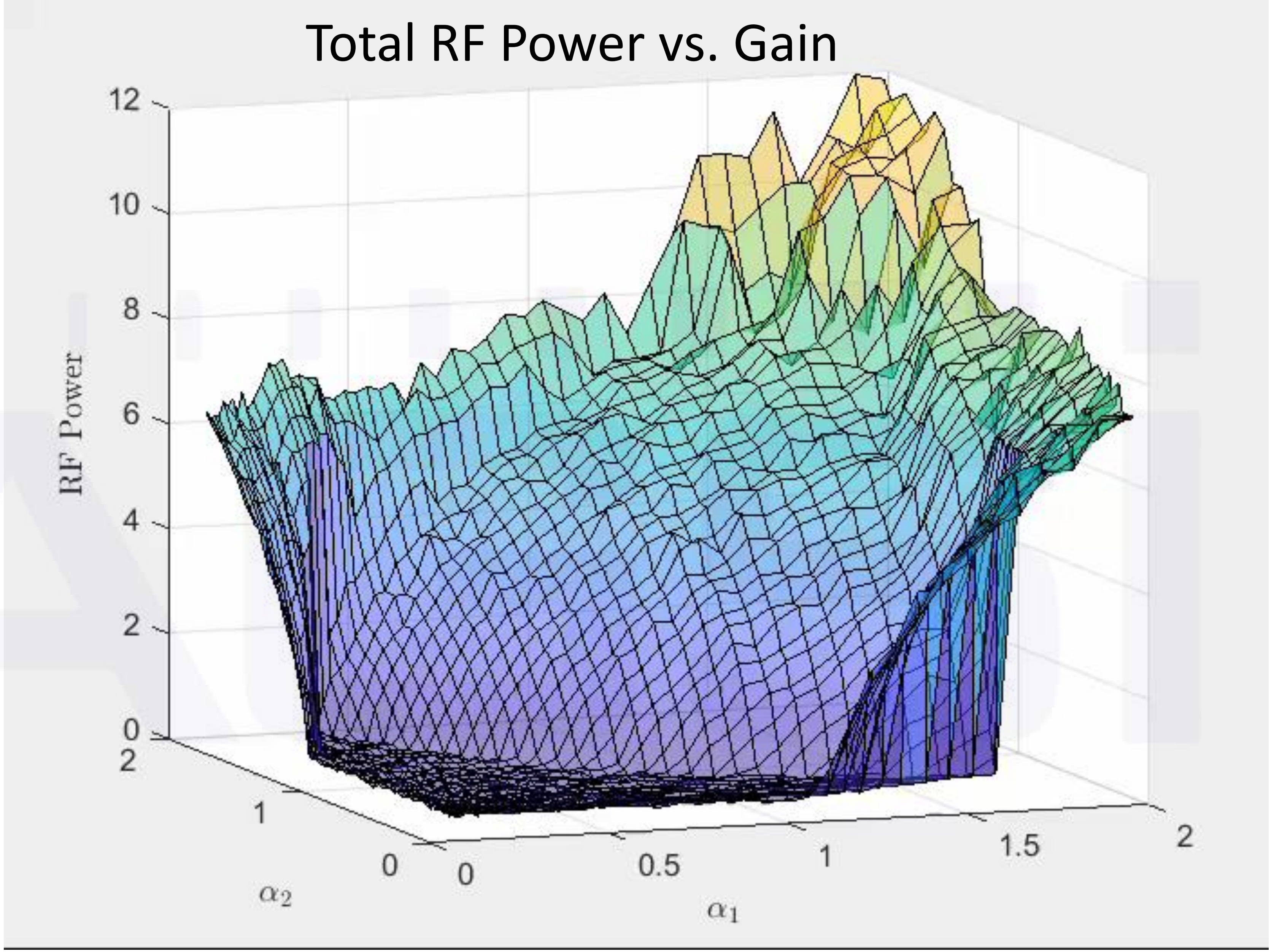
$\alpha_2 \propto$ Optical Power 2

3. Exp. Results

Balanced

Experiment

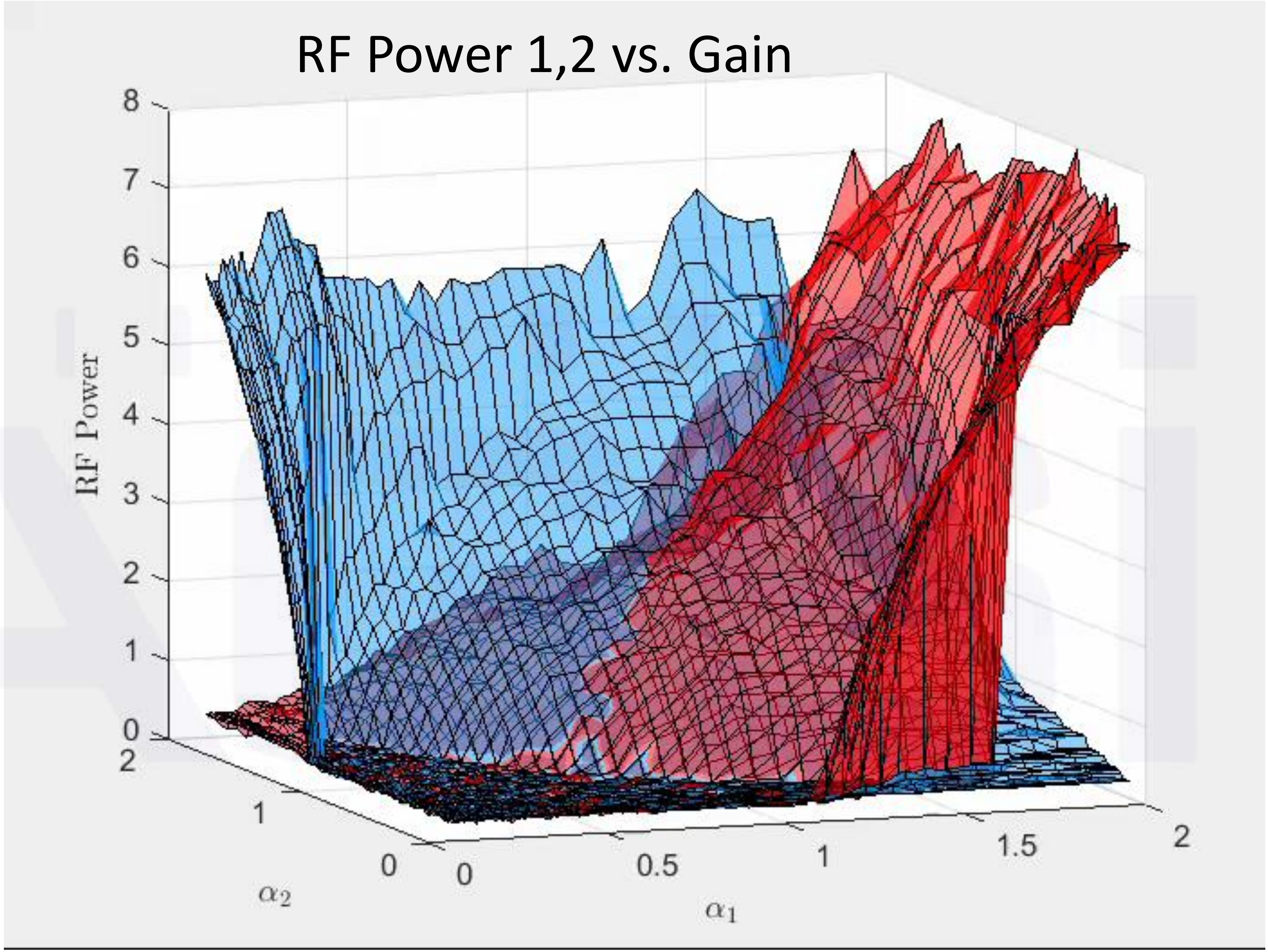
$\alpha_1 \propto$ Optical Power 1
$\alpha_2 \propto$ Optical Power 2



3. Exp. Results

Balanced

Experiment



$\alpha_1 \propto$ Optical Power 1
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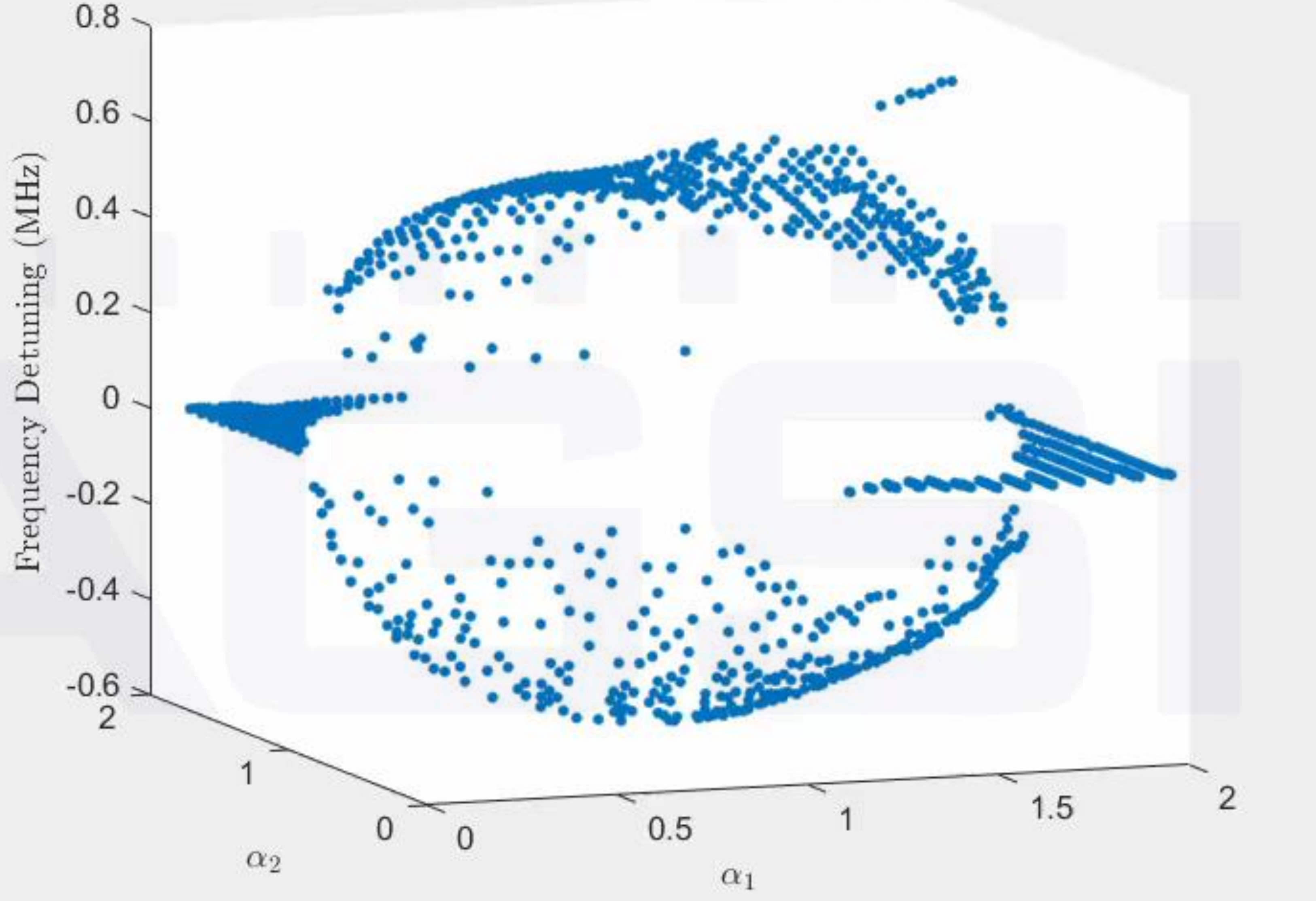
3. Exp. Results

Balanced

Experiment

Frequency Detuning vs. Gain



$\alpha_1 \propto$ Optical Power 1
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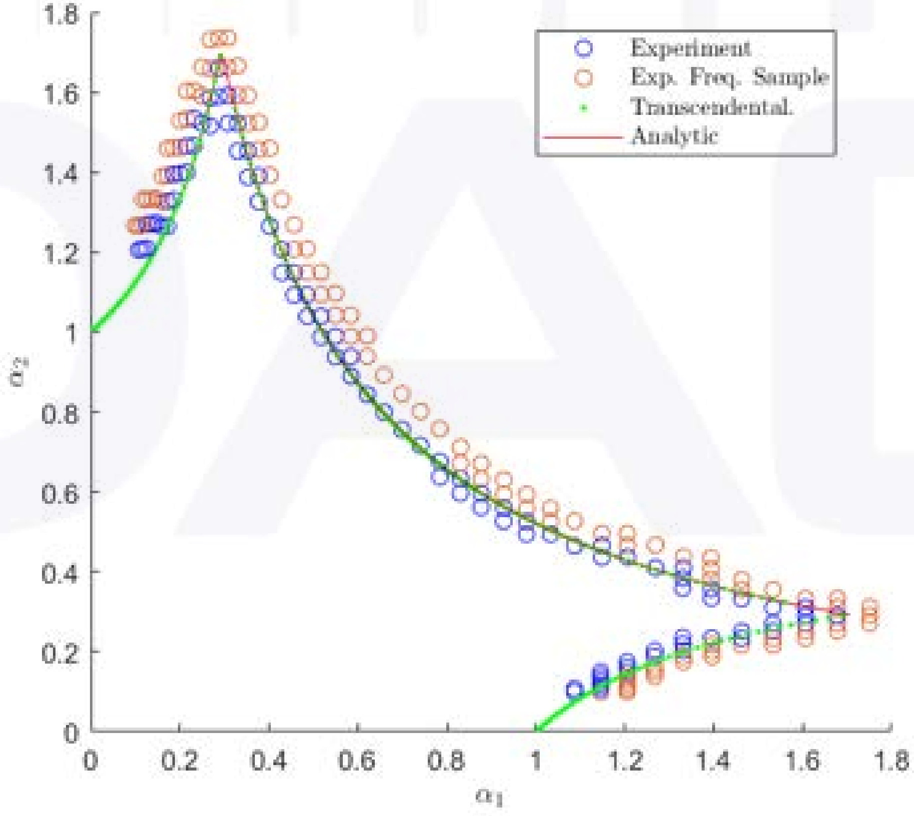
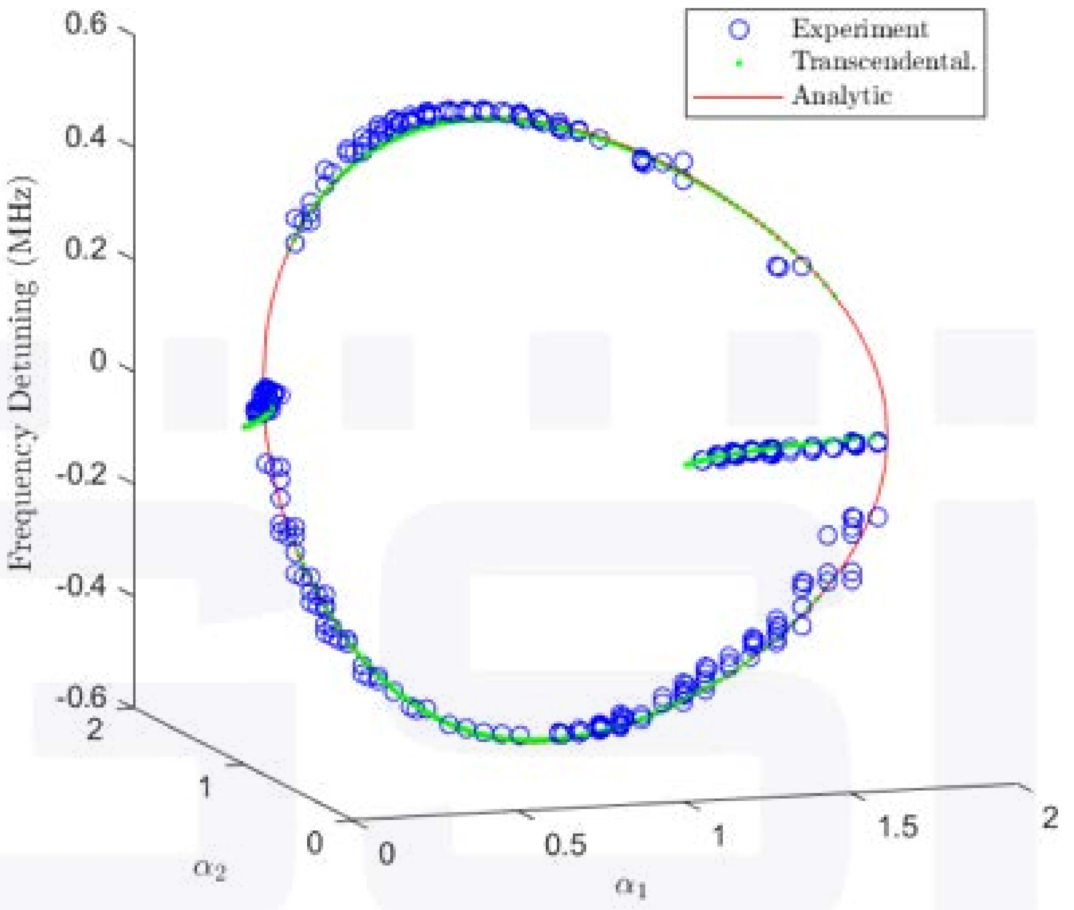
3. Exp. Results

Experimental

Balanced Boundary





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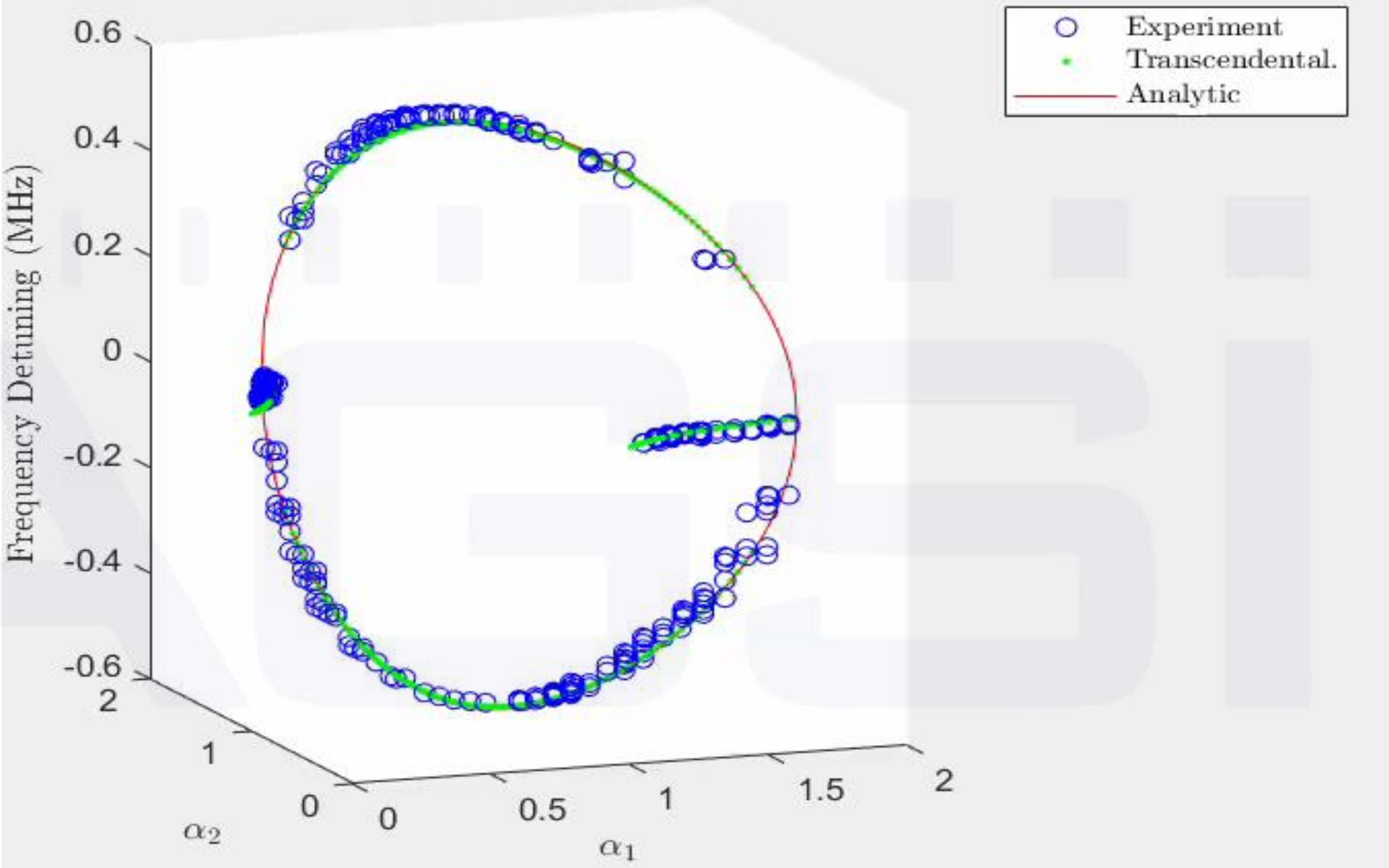
3. Exp. Results

Experimental

Balanced Boundary

Frequency Detuning vs. Gain



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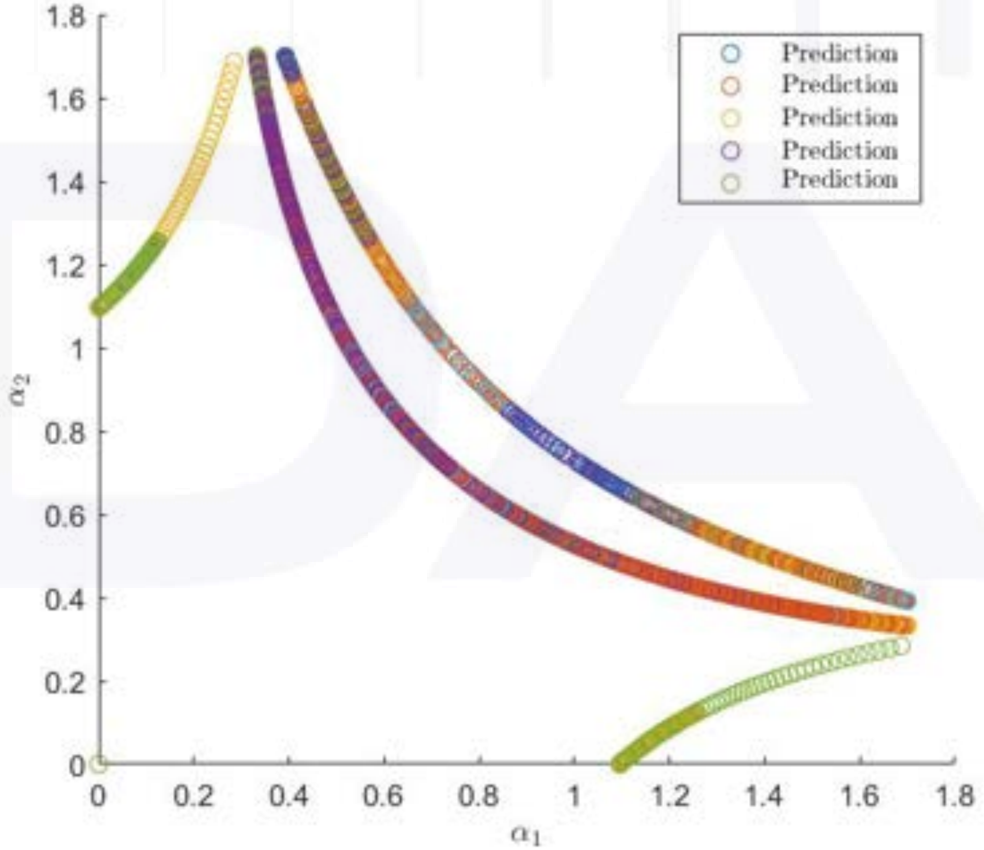
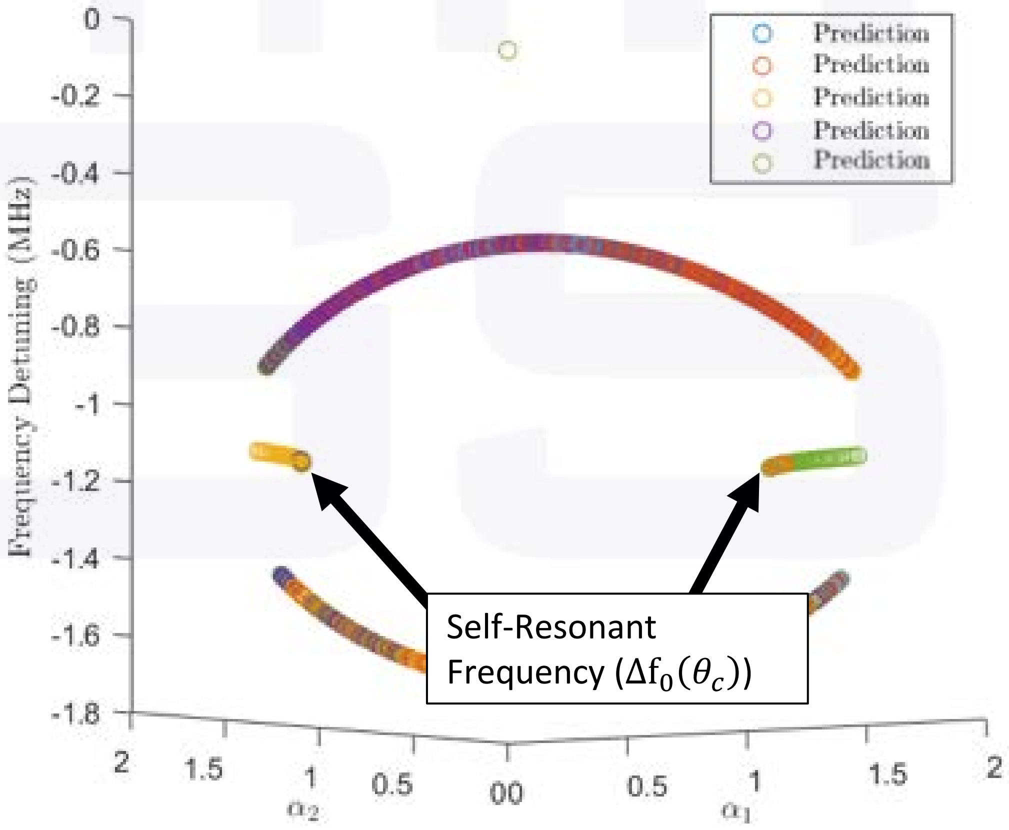
**Some Definitions

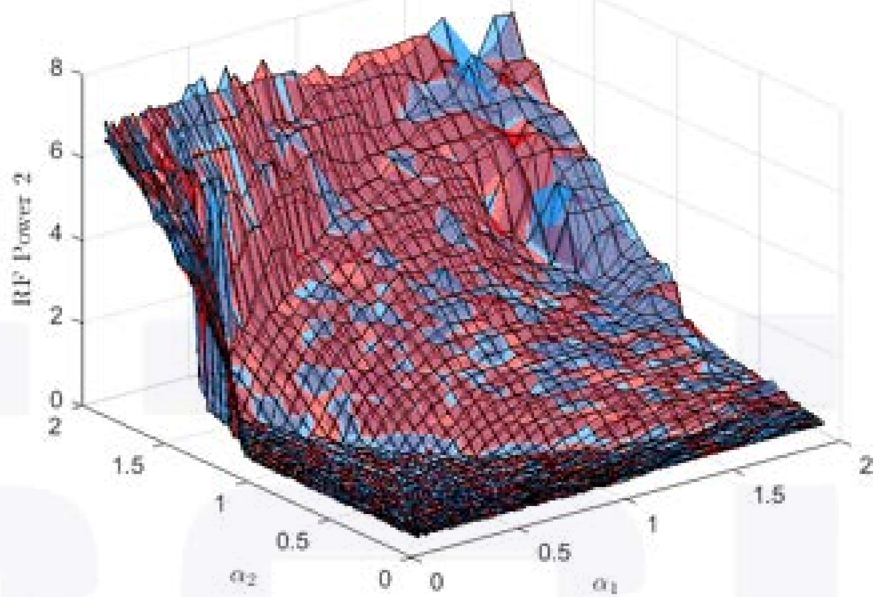
Phase Matched by $\theta_c = \pi/2$ (Imbalanced)

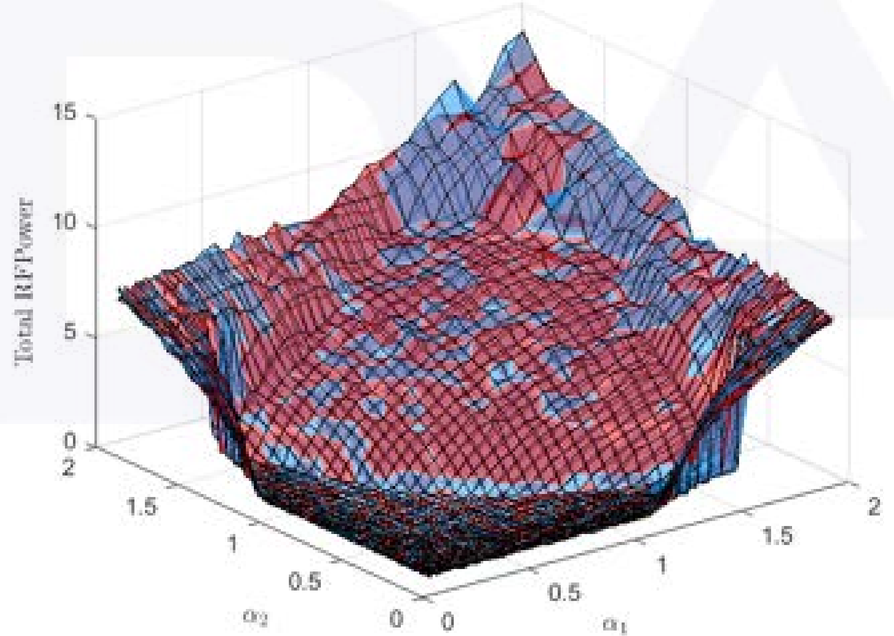


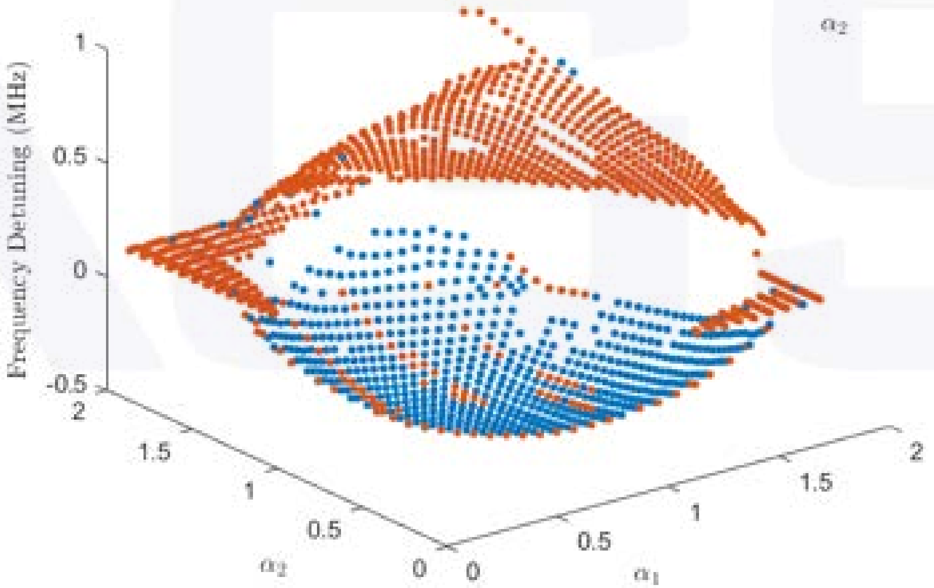


3. Exp. Results

Phase Matched
($\Delta f_0(\theta_c) = 0.93$ MHz)
Experiment



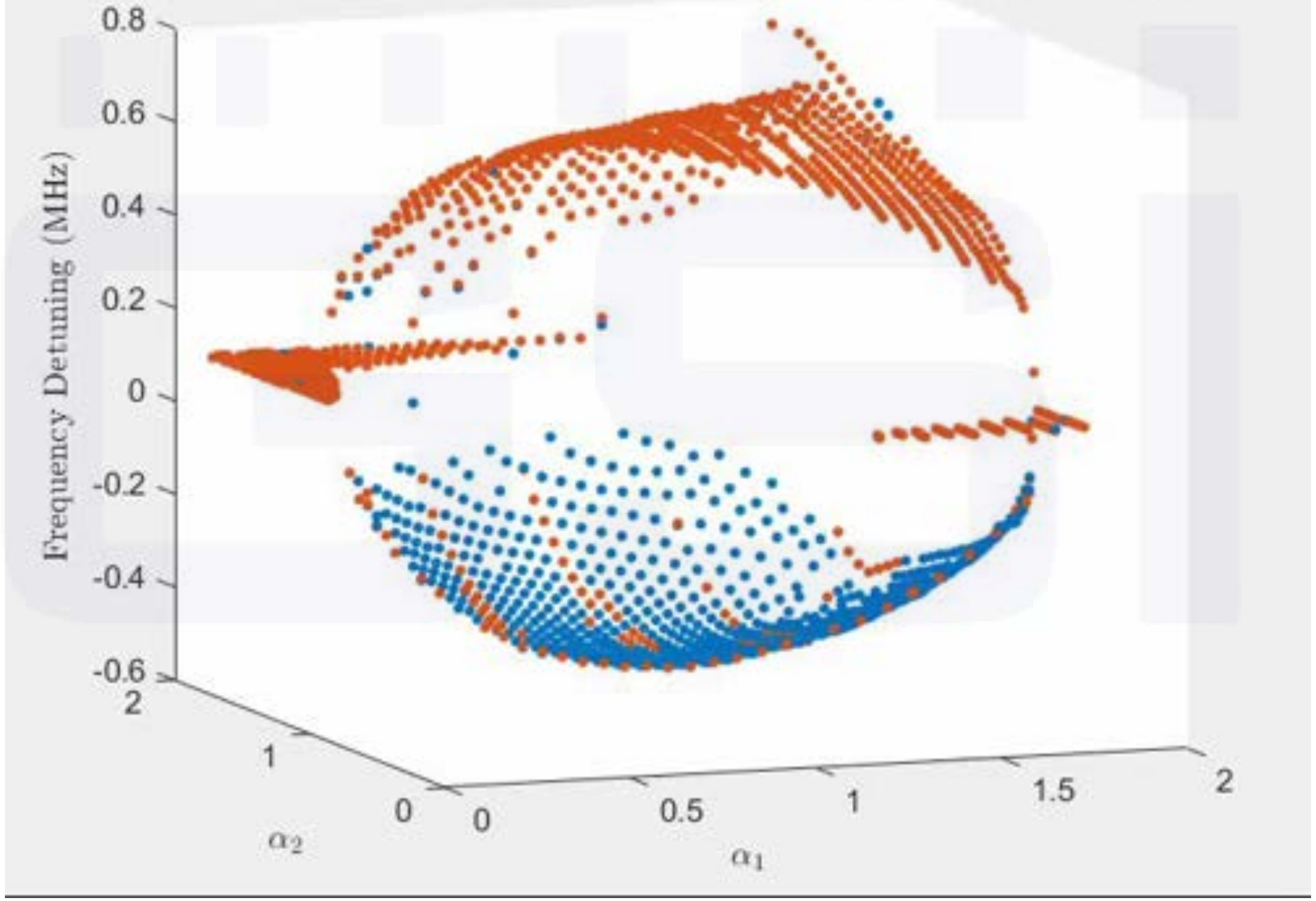




3. Exp. Results

Phase Matched $(\Delta f_0(\theta_c) = 0.93\ \text{MHz})$ Experiment

Frequency Detuning vs. Gain



$\alpha_1 \propto$ Optical Power 1
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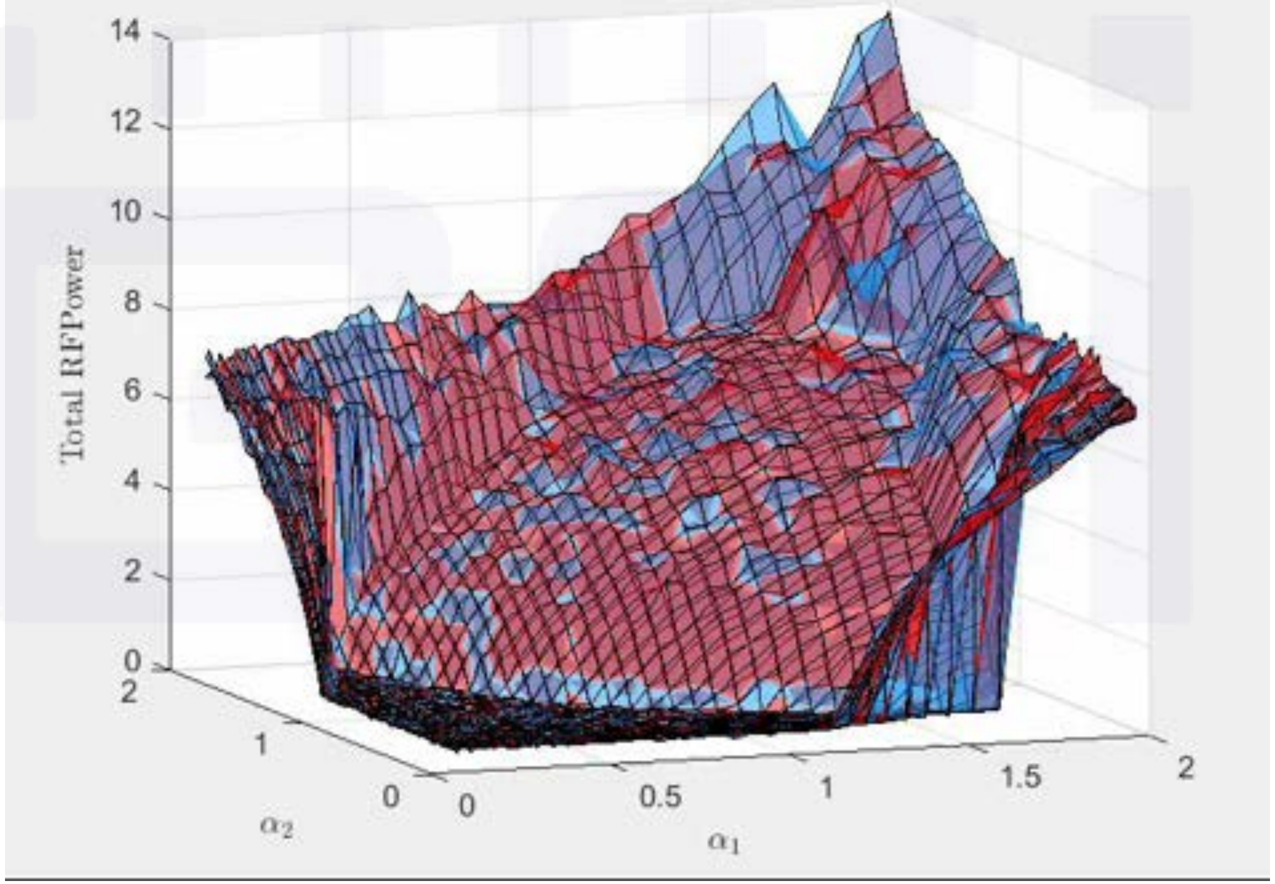
3. Exp. Results

Phase Matched $(\Delta f_0(\theta_c) = 0.93\text{ MHz})$ Experiment

RF Power vs. Gain



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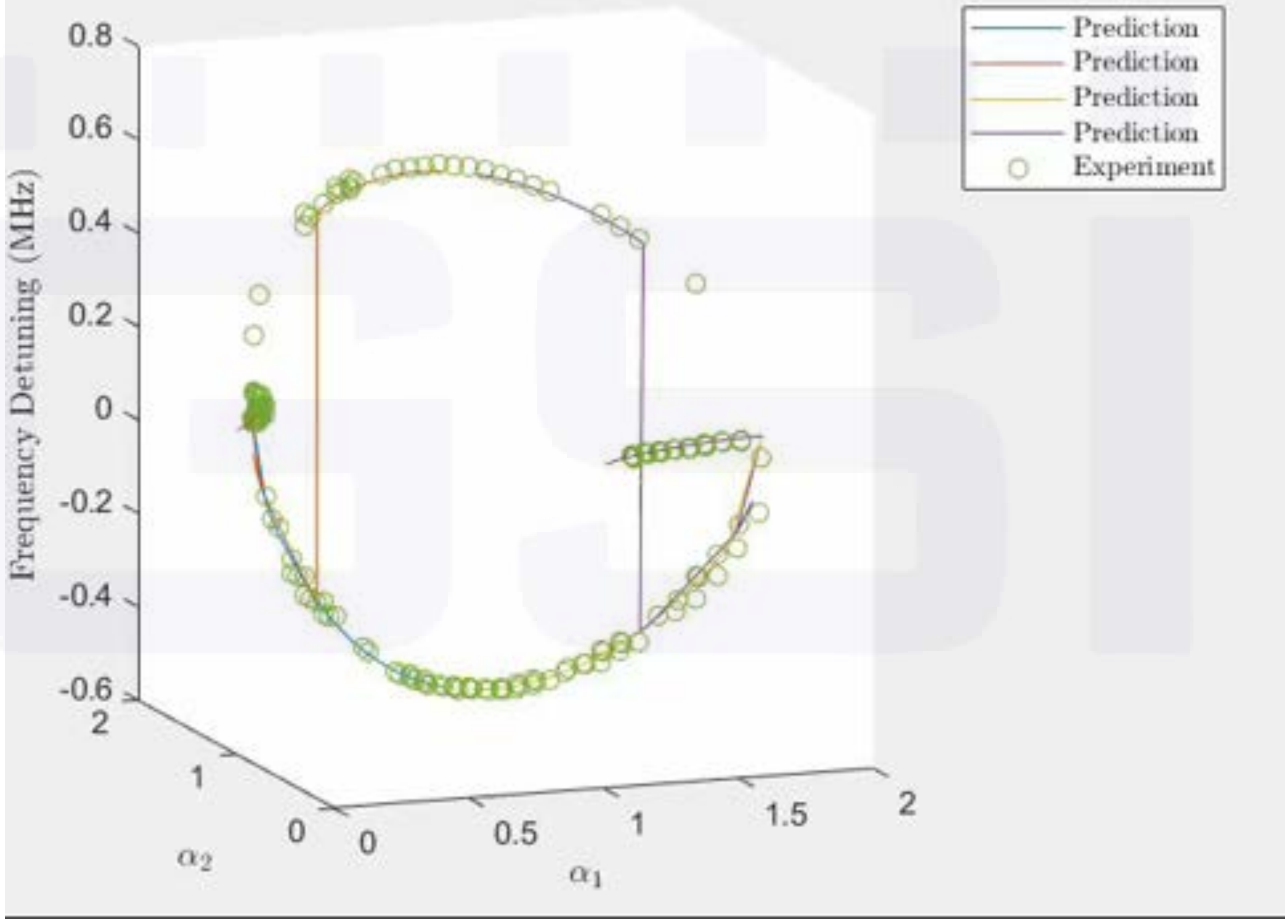
Frequency Detuning vs. Gain



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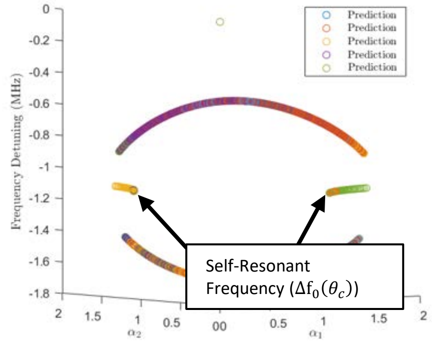
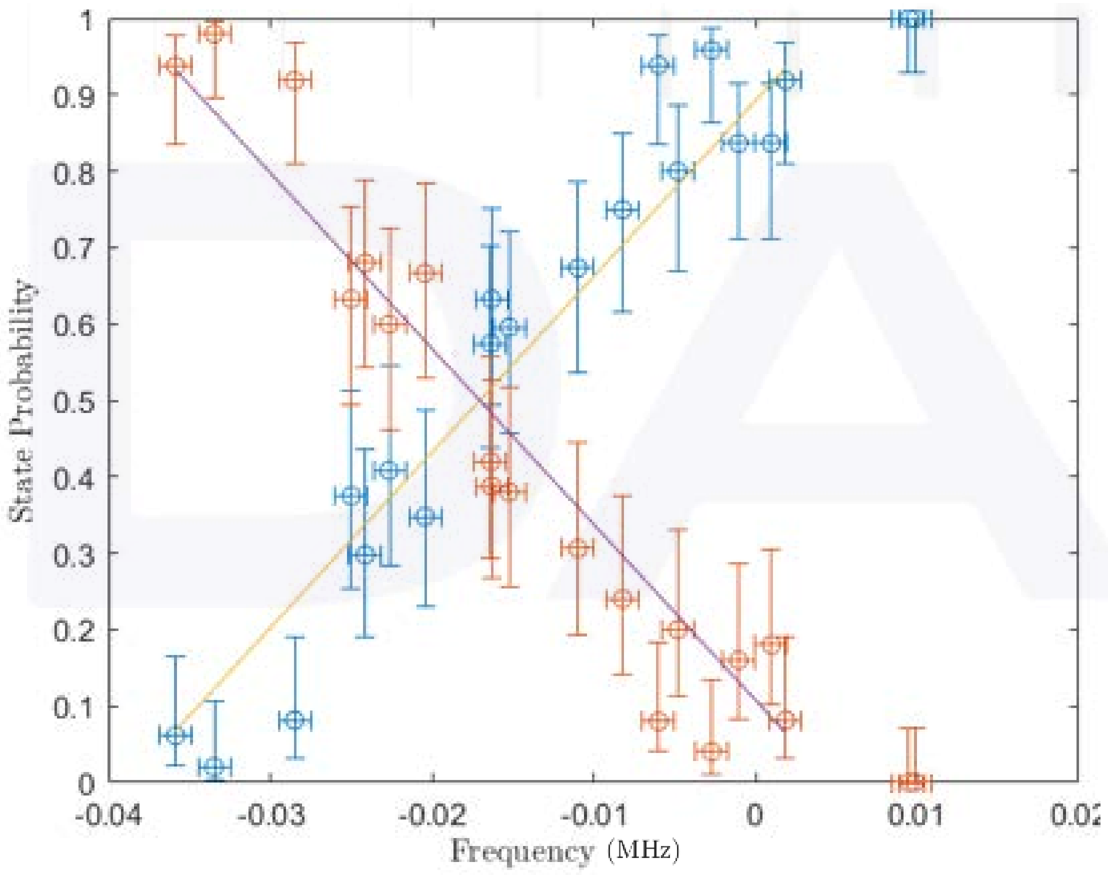
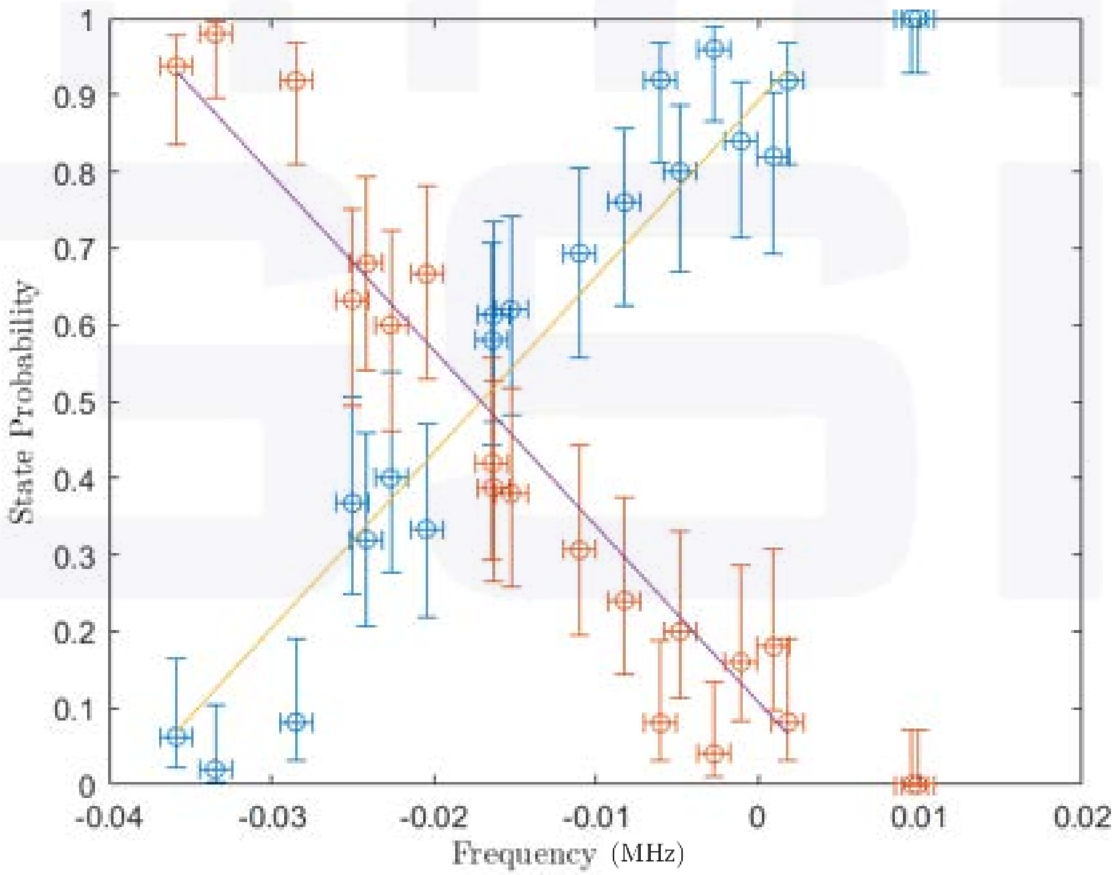
3. Exp. Results



State Probability vs. Self-Resonant Frequency ($\Delta f_0(\theta_c)$)





4. Conclusion

- We were able to construct a coupled OEO model with algebraic/analytic approximate solutions (predictions)
  - Utilized the slowly varying envelope approach
  - Transcendental condition for the first bifurcation
  - Exceptional line constant for $\theta_d = 0$
  - First bifurcation algebraic approximation (balanced: $\theta_d, \theta_c = 0$)
    - Convergent in both limits!
  - For $\theta_d = 0$, $\omega_{th}^2 = \frac{\Delta^2}{4}(2\alpha_2\alpha_1 - 1)$ [it's a hyperboloid!]
- Bifurcation boundary matches theory well for all time delays/phases
- Both threshold solutions are available for unbalanced
  - Hysteresis
- We found a states exhibit a probabilistic dependency on self-resonant frequency