Temporal Convergence and Stability Assessment of the Generalized Finite Element Method for Multi-Scale Field Problems

Student: TJ Miller

Student Email: Miller.8306@buckeyemail.osu.edu

Faculty: Jack McNamara

Faculty Email: Mcnamara.190@osu.edu

AFRL Sponsor: Pat O'Hara

AFRL Directorate: AFRL/RQ

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Objectives

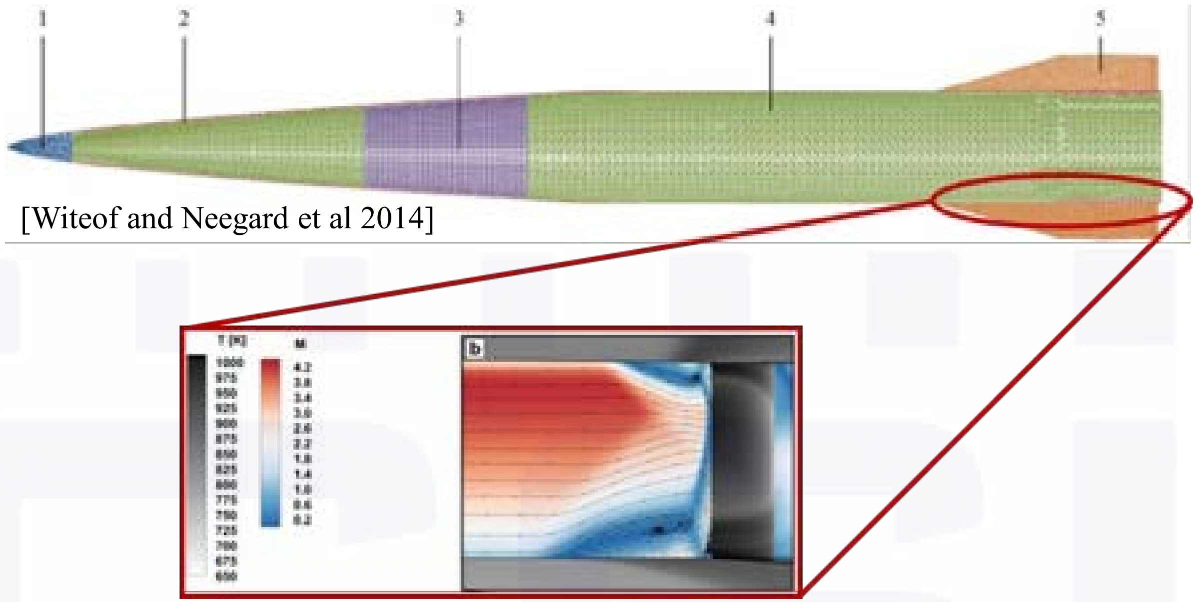
Goal: Expand research on multi-scale numerical modeling strategies to transient heat transfer problems with highly localized loading conditions through establishing implementation strategies and characterizing the temporal convergence behavior of GFEM

1. Assess the temporal convergence and accuracy of GFEM applied to a model problem in multiple dimensions
2. Assess temporal stability of the GFEM in heat transfer analysis
3. Investigate computational savings compared to standard methods
4. Shed light on nuances and implementation strategies of the GFEM

Challenges with Hypersonic Analysis

- Modern day engineering problems in hypersonic vehicle design are dominated by heat transfer
  - Fine-scale and transient loading conditions
  - Coupled, multi-physic interactions
- Current multi-scale modeling strategies lack power to resolve all spatial-temporal scales on a global level
  - Fine meshes needed for spatial gradients
  - Broad regions of refinement for transient features
  - Small time steps to maintain temporal stability
- High-fidelity solutions often require large amounts of CPU power, time and memory

Question: How can we simultaneously capture fine-scale features and global phenomena within a multi-physics simulation?



[Witeof and Neegard et al 2014]

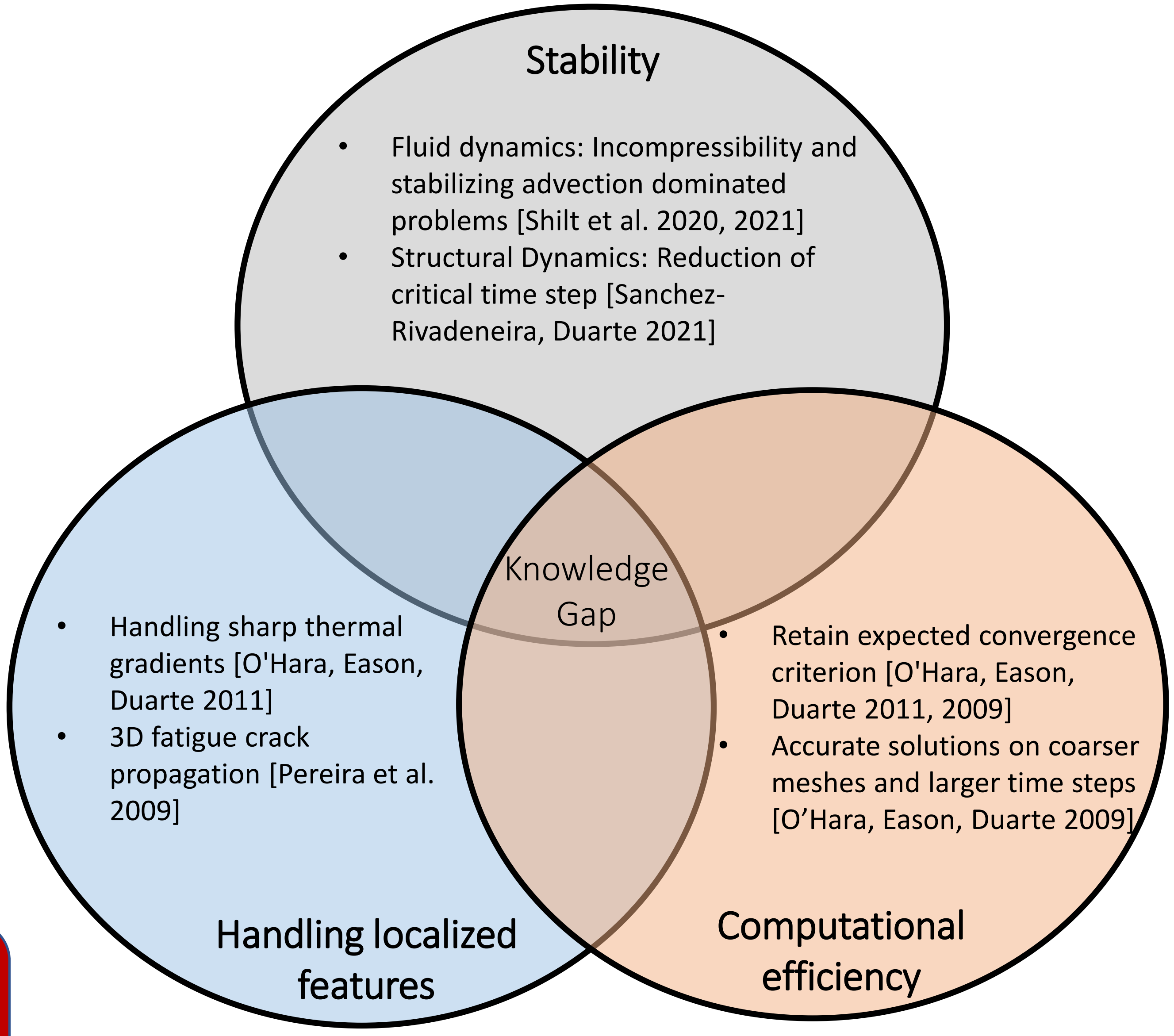
Image courtesy if Jon Willem, The Ohio State University

- Above is the flow and temperature field over a torque tube experiencing sharp thermal gradients on a small scale due to shock-boundary layer interactions
- Accurately resolving local features across all spatial-temporal scales, while avoiding local mesh refinement and advanced multi-scale methods, is essential for practical modeling of multi-physic simulations

Motivation

- Heat transfer in extreme environments is multi-scale and coupled with fluids and structural analysis
- Mathematically, heat transfer is a scalar equation
  - **Implementation differs from vectoral analysis of fluids and tensorial analysis of structures**
- Enabling solutions of heat transfer problems in extreme conditions is essential for high-speed vehicle design
- The GFEM incorporates solution-tailored shape functions to alleviate the need for local mesh refinement
- Current work has focused on ability for GFEM to capture localized features efficiently and stabilization of fluids and structural problems in multi-scale environments
- Lack of research to extend these concepts to heat transfer analysis leads to a gap in knowledge

Hypothesis: GFEM can enable high-fidelity solutions of extreme multi-scale heat transfer problems that are currently prohibitive in the context of multi-physics simulations



Overview of the GFEM

Modifies FEM framework to introduce solution tailored "enrichments" into standard FEM space

FEM Approximation Space

$$\{\varphi_\alpha(x)\}_{\alpha=1}^{N}$$

$*$

Enrichment Space

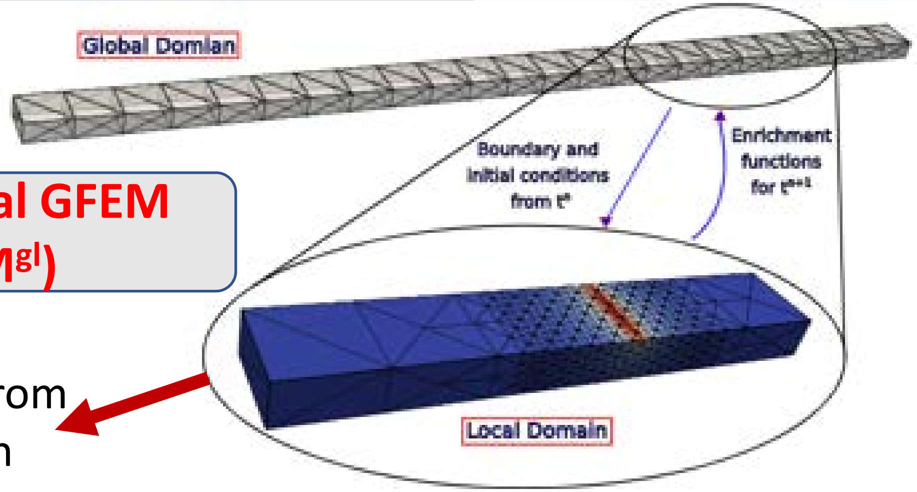
$$\{L_{\alpha j}(x)\}_{j=1}^{m_\alpha}$$

- Can be any function; typically derived from *a priori* knowledge

- GFEM shape functions formed through a product of standard FEM shape functions and enriched trial space

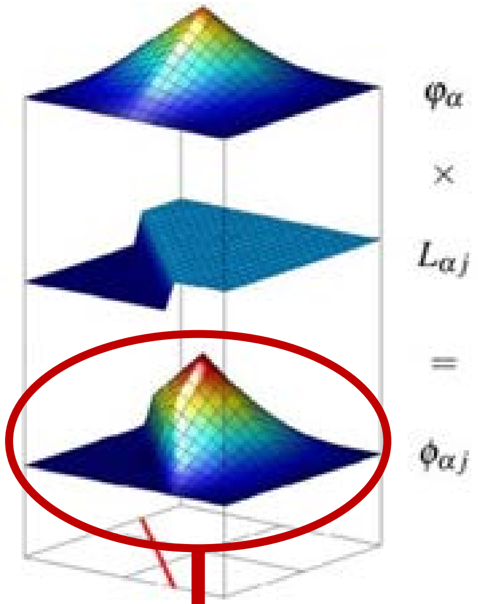
GFEM Enrichment Space

$$\phi_{\alpha j} = \{\varphi_\alpha(x) L_{\alpha j}(x)\}_{j=1}^{m_\alpha}$$



Global-Local GFEM ($GFEM^{gl}$)

Enrichments extracted from local problem solution



Includes discontinuity into shape function

Final approximation: $u^h(x) = \sum_{\alpha \in I_h} \hat{u}_\alpha \varphi_\alpha(x) + \sum_{\alpha \in I_h} \varphi_\alpha(x) \sum_{j=1}^{m_\alpha} \tilde{u}_{\alpha j} L_{\alpha j}(x)$

GFEM elegantly handles fine-scale features by directly introducing these features into the computational domain, alleviating local mesh refinement

Challenges with GFEM

GFEM shape functions inherently linearly dependent

- Singular matrices, even after boundary condition application →Leads to difficulties in solving the system and potential errors
- Ill-conditions matrices due to close to singular matrices → Leads to errors in the solution

Implementation rarely straightforward

- Difficult to apply boundary conditions due to enrichments → Enrichments not necessarily zero at the nodes
- *a priori* information not always available →Difficult to derive proper enrichment functions

Need: Development of GFEM to confidently expand its use cases and identify technical challenges that must be overcome

Beam Subjected to a Sharp Heat Flux

Property Definitions	
ρc	$18.3 \frac{ft * lbf}{in^3 * °F}$
k	$2.92 \frac{ft * lbf}{in * °F * s}$
y_0	$0.25 in$
x_0	$8.8 in$
I_0	$295.03 \frac{ft * lbf}{s}$
a	$0.025 in$
γ	$10 \frac{1}{s}$
h	$11 \frac{ft * lbf}{in^2 * °F * s}$
u_∞	$70°F$

$$\rho c \frac{\partial u}{\partial t}(\boldsymbol{x}, t) = k \nabla^2 u(\boldsymbol{x}, t) \; in \; \Omega$$

$$-k \frac{\partial u}{\partial n} = h(u(x,t) - u_\infty) \; on \; \Gamma_c = \partial\Omega \setminus \Gamma_c$$

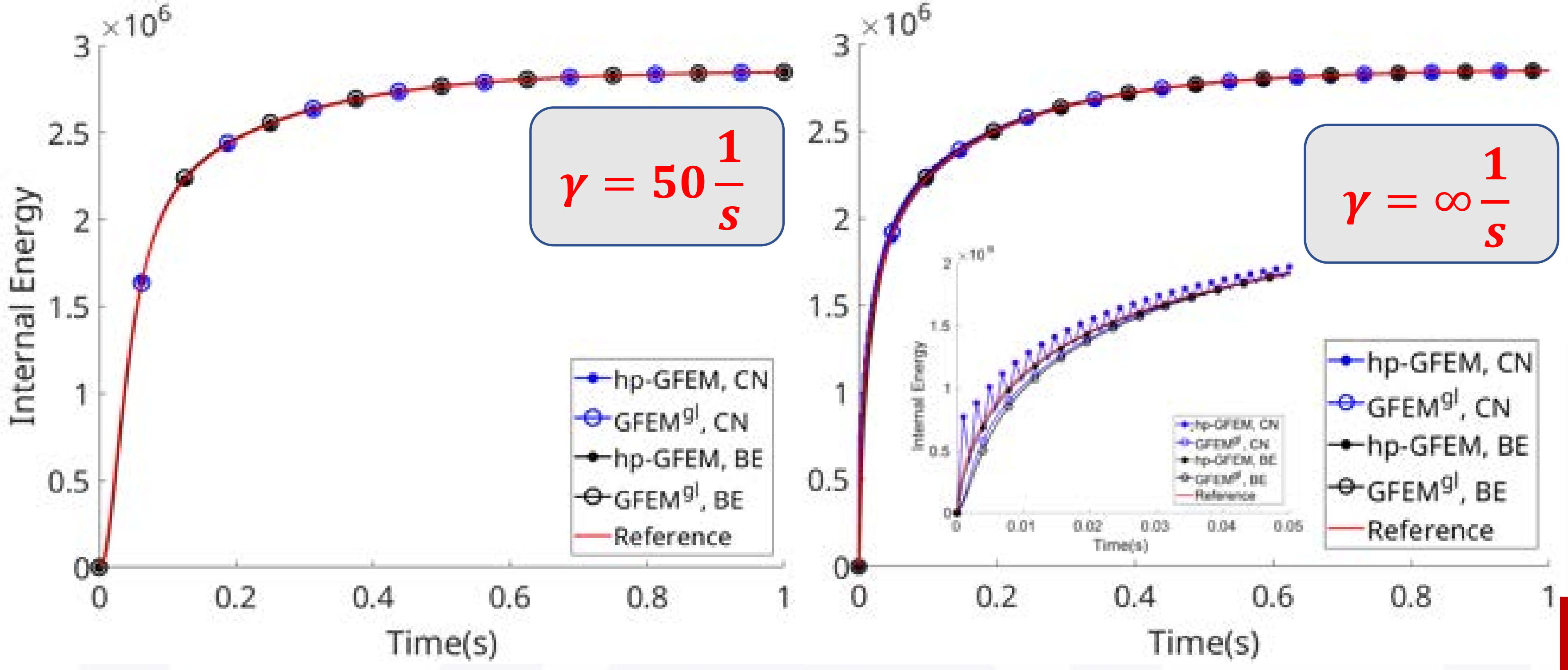
$$\Omega = \{0 < x < 12in, 0 < y < 0.5in, 0 < z < 0.24in\}$$

$$-k \frac{\partial u}{\partial n} = \frac{I_0}{2\pi a^2} e^{-\frac{\left(x - x_{front}(t) + \beta(y - y_0)\right)^2}{2a^2}} (1 - e^{-\gamma t}) \;\; on \;\; \Gamma_n$$

$$x_{front} = x_0 + Vt$$

$$\Gamma_n = \{8in < x < 10in, 0 < y < 0.5in, z = 0.24in\}$$

Time Evolution of Internal Energy for Different γ



Enrichments:

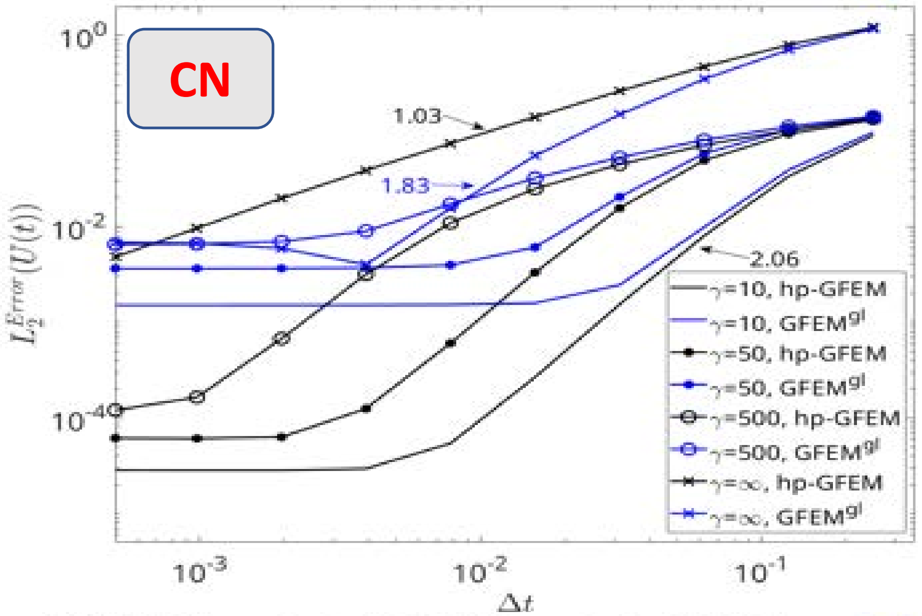
$$L_{\alpha j} = \left\{ \left(\frac{x-x_\alpha}{h}\right)^l \left(\frac{y-y_\alpha}{h}\right)^m \left(\frac{z-z_\alpha}{h}\right)^k \right\}^p_{l,m,k=0}$$

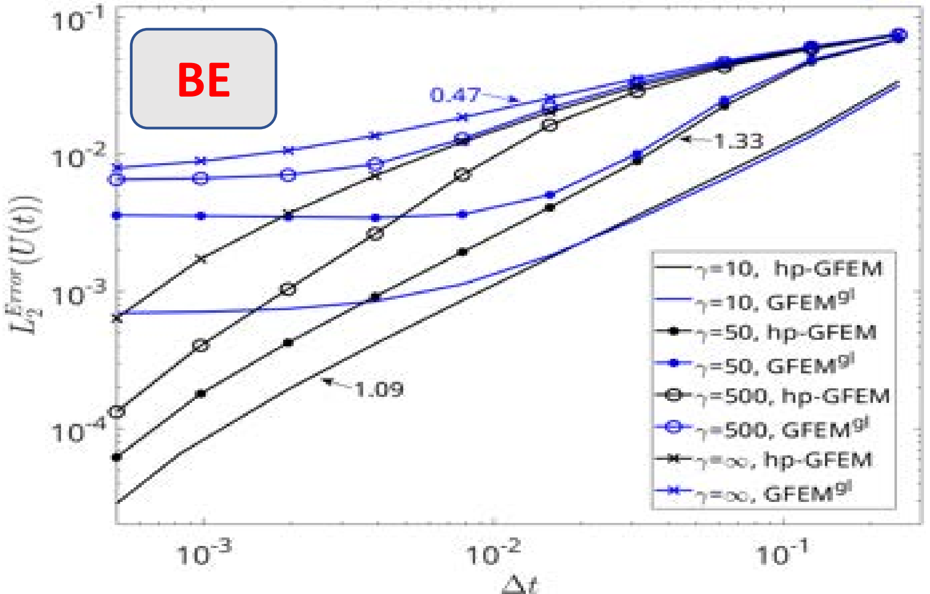
$$k \le m + l$$

	hp-GFEM	GFEM^{gl}
Degrees of Freedom	913,320	420/ 104,090 (Global/Local)
Enrichment Order	$p = 3$	$p = 2$+local/ $p = 2$
Number of Time-steps	1024	1024

- Energy evolution prediction depends on ability the time-integrator and spatial approximation space to capture the sharp temporal gradient
  - Low order approximations exhibit **oscillations** in Crank-Nicolson (CN) solutions when γ is large
  - **Time-dependent shape functions** accurately predict the internal energy over all cases
- **GFEM^{gl} enables accurate prediction of sharp temporal gradients for multiple time-integrators**

Temporal Convergence of GFEM with Different Time Integrators and γ





- Shown left are accurate multi-scale solutions with **larger time steps**
- GFEM using **solution-tailored, exponential enrichments** obtains temporal convergence
- $GFEM^{gl}$ curves **match** hp-GFEM until error saturates for Backward Euler (BE)
- $GFEM^{gl}$ achieves **optimal convergence** where hp-GFEM doesn't for CN
  - Evident for $\gamma = \infty$ as hp-GFEM convergence rate has reduced to 1
- **GFEM enables temporally convergent multi-scale solutions for multiple time integrators and configurations**

Conclusions and Future Work

- Temporal convergence obtained with time-dependent, solution tailored enrichments on coarse grids
- Convergence study demonstrates GFEM can achieve accurate solutions on coarse meshes and larger time steps
- Solution-tailored, time-dependent enrichments damped numerical oscillations and recovered theoretical convergence rates
- Results provide improved confidence in the ability of GFEM to enable simulations of design critical multi-scale, multi-physics problems of hypersonic systems
- Initial stability study (not shown) and convergence results indicate potential for GFEM to increase critical time steps with solution-tailored enrichments

- Future Work
  - Study on the Forward Euler method
  - Conduct an in-depth stability analysis to determine how the choice of approximation effects critical time steps